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ON A CONJECTURE FOR THE IDENTITIES IN MATRIX ALGEBRAS WITH INVOLUTION [1]

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In a previous paper the author made a conjecture on the minimal degree $4n$ of the polynomials, which are identities for the matrix algebra of order $2n$ with symplectic involution considered as polynomials both in symmetric and skew-symmetric due to the involution variables.
In the present paper we establish that the conjecture is not true at least for the case of the matrix algebra of fourth order by giving an example of such an identity of degree seven, which is a Bergman type identity.
For the matrix algebra of sixth order with symplectic involution we describe the class of all Bergman type identities both in symmetric and skew-symmetric variables of minimal degree (which appeared to be 14). For arbitrary polynomials being identities of the considered type the question of their minimal degree is still open.

Keywords: Bergman type polynomials, symplectic involution, *-identities in symmetric or skew-symmetric variables

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Let K be a field of characteristics zero with elements α, β, etc. We call Bergman type polynomials the following class of homogeneous polynomials introduced by Formanek [2] and Bergman [1] for investigating matrix identities by means of commutative algebra.

To a homogeneous polynomial in commuting variables

$$g(t_1, \ldots, t_{n+1}) = \sum \alpha_p t_1^{p_1} \ldots t_{n+1}^{p_{n+1}} \in K[t_1, \ldots, t_{n+1}] \tag{1}$$

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we relate a polynomial $v(g)$ from the free associative algebra $K\langle x, y_1, \ldots, y_n\rangle$,

$$v(g) = v(g)(x, y_1, \ldots, y_n) = \sum \alpha_p x^{p_1} y_1 \ldots x^{p_n} y_n x^{p_{n+1}}. \tag{2}$$

Any homogeneous and multilinear in $y_1, \ldots, y_n$ polynomial $f(x, y_1, \ldots, y_n)$ can be written as

$$f(x, y_1, \ldots, y_n) = \sum_{i=(i_1,\ldots,i_n)\in Sym(n)} v(g_i)(x, y_{i_1}, \ldots, y_{i_n}), \tag{3}$$

where $g_i \in K[t_1, \ldots, t_{n+1}]$.

We consider Bergman type polynomials on subalgebras of the matrix algebra $M_{2n}(K, *)$ with symplectic involution defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}, \tag{4}$$

where A, B, C, D are $(n \times n)$-matrices and t is the usual transpose.

Details on polynomial and $*$-polynomial identities one may find in [6, 4].

For an algebra R with involution $*$ we have $(R, *) = R^+ \oplus R^-$, where $R^+ = \{r \in R \mid r^* = r\}$ and $R^- = \{r \in R \mid r^* = -r\}$. Let $K\langle X\rangle$ be the free associative algebra. We call $f(x_1, \ldots, x_m) \in K\langle X\rangle$ a $*$-polynomial identity for the algebra $(R, *)$ both in symmetric and skew-symmetric variables if $f(r_1^+, \ldots, r_m^+) = 0$ for all $r_1^+, \ldots, r_m^+ \in R^+$ and $f(r_1^-, \ldots, r_m^-) = 0$ for all $r_1^-, \ldots, r_m^- \in R^-$.

One of the reasons to study this kind of identities is the following. The algebra R^- is a Lie algebra with respect to the new multiplication $[r_1^-, r_2^-] = r_1^- r_2^- - r_2^- r_1^-$, $r_1^-, r_2^- \in R^-$, and the identities in skew-symmetric variables for $(R, *)$ are weak polynomial identities for the pair (R, R^-), i.e. the identities of the related representation of R^-. Similarly, R^+ is a Jordan algebra with respect to the multiplication $r_1^+ \circ r_2^+ = r_1^+ r_2^+ + r_2^+ r_1^+$, $r_1^+, r_2^+ \in R^+$, and the identities in symmetric variables are weak polynomial identities for the pair (R, R^+). In this way, the identities both in symmetric and skew-symmetric variables are the weak identities which hold for both pairs (R, R^-) and (R, R^+).

In [5] we have discussed the minimal degree of such identities and have made the following conjecture:

Conjecture [5, Conjecture 3.1]. The minimal degree of a $*$-identity both in symmetric and skew-symmetric variables in $M_{2n}(K, *)$ for $n \geq 2$ is equal to $4n$.

In the present paper we show that *this conjecture is not true* (at least for $n = 2$). Considering the matrix algebra $M_4(K, *)$, we give an example of a Bergman type polynomial, which is a $*$-identity both in symmetric and skew-symmetric variables of degree 7.

For the matrix algebra of sixth order with symplectic involution we describe the class of all Bergman type identities both in symmetric and skew-symmetric variables of degree 14, which is their minimal degree.

These considerations are consequences of the following main theorem:

Theorem 1. *Any Bergman type identity in skew-symmetric variables for the algebra $M_{2n}(K,*)$ with symplectic involution is a $*$-identity in symmetric variables as well.*

For proving the theorem we need some preliminary results.

Proposition 1. [4, Theorem 1] *Let a polynomial $f(x,y_1,\dots,y_n)$ of type (3) be a $*$-identity in skew-symmetric variables for $M_{2n}(K,*)$. Then the polynomial*

$$\prod_{\substack{1\le p<q\le n+1\\ (p,q)\neq(1,n+1)}} (t_p^2-t_q^2)(t_1-t_{n+1})$$

divides the polynomials g_i from (1) for all $i=(i_1,\dots,i_n)$.

Lemma 1. *Any generic symmetric matrix (with respect to the symplectic involution) is diagonalizable.*

Proof. It is a well-known fact [6, Theorem 2.5.10] that a $(2n\times 2n)$-matrix x, symmetric with respect to the symplectic involution, satisfies an equation $p(x)=0$ of degree n and its characteristic polynomial is $p^2(x)$. Thus any generic symmetric matrix has at most n different characteristic values. A generic symmetric matrix a of order $2n$ has the following presentation in the form (4): A is a generic $(n\times n)$-matrix, D is its transpose, and B and C are generic skew-symmetric $(n\times n)$-matrices. If it has less than n different characteristic values, then the same will hold for any generic symmetric matrix and also for

$$x=\sum_{i=1}^{n}\rho_i(e_{ii}+e_{n+i,n+i}), \tag{5}$$

where ρ_i are algebraically independent variables. This is a contradiction, because x has n different characteristic values. Thus any generic symmetric matrix is diagonalizable and we may consider x in the form (5).

Lemma 2. *For any polynomial $g(t_1,\dots,t_{n+1})$ of type (1) divisible by the product $\prod_{i<j}(t_i-t_j)$, the associated polynomial $f(x,y_1,\dots,y_n)$ of type (3) vanishes on $M_{2n}(K,*)$ for x being symmetric due to the involution and for arbitrary $y_1,\dots,y_n\in M_{2n}(K,*)$.*

Proof. According to Lemma 1, the matrix x can be replaced by $\bar{x}=\sum_{i=1}^{n}\rho_i(e_{ii}+e_{n+i,n+i})$. The linearity of f in $y_1,\dots,y_n$ allows to consider the variables $y_i, i=1,\dots,n$, as matrix units from $M_{2n}(K,*)$ with nonzero product, namely $\bar{y_i}=e_{p_ip_{i+1}}$ for $p_i=1,\dots,2n$. Thus we get

$$f(\bar{x},\bar{y_1},\dots,\bar{y_n})=v(g)(\bar{x},\bar{y_1},\dots,\bar{y_n})=g(\bar{\rho}_{p_1},\dots,\bar{\rho}_{p_n},\bar{\rho}_{p_{n+1}})e_{p_1p_{n+1}},$$

where $\bar{\rho}_p = \rho_p$ if $p \le n$, and $\bar{\rho}_p = \rho_{p-n}$ if $n+1 \le p \le 2n$. Since $\bar{\rho}_{p_i} \in \{\rho_1, \ldots, \rho_n\}$, $i = 1, \ldots, n+1$, we have $\bar{\rho}_{p_i} = \bar{\rho}_{p_j}$ for some $i \ne j$. This means that $g(\bar{\rho}_{p_1}, \ldots, \bar{\rho}_{p_n}, \bar{\rho}_{p_{n+1}}) = 0$ as $t_i - t_j$ divides the polynomial $g(t_1, \ldots, t_{n+1})$.

Proof of Theorem 1. Let f be a Bergman type identity in skew-symmetric variables. According to Proposition 1, the considered polynomial in commuting variables is divisible by the product $\prod_{i<j}(t_i - t_j)$, $1 \le i < j \le n+1$. Lemma 2 gives that f is a Bergman type identity in symmetric variables as well.

It is known [3, pp. 318–319] that $[[x_1^2, x_2]^2, x_1] = 0$ is a $*$-identity of minimal degree in skew-symmetric variables for $M_4(K,*)$.

The following proposition illustrates Theorem 1.

Proposition 2. *The linearization in x_2 of the polynomial $[[x_1^2, x_2]^2, x_1]$ is a Bergman type $*$-identity for $M_4(K,*)$ both in symmetric and skew-symmetric variables.*

Proof. The proposition follows immediately from Theorem 1, because $[[x_1^2, x_2]^2, x_1]$ is an identity in skew-symmetric variables for $M_4(K,*)$ and its linearization in x_2 is of Bergman type. We shall give an alternative proof and shall show that it is a consequence of an identity of special form.

First, for a Bergman type identity in symmetric variables of degree n we describe the general form of its consequence in symmetric variables of degree $n+1$.

Let

$$\begin{aligned} f(x, y_1, y_2) &= v(g_{(1,2)})(x, y_1, y_2) + v(g_{(2,1)})(x, y_2, y_1) \\ &= v(g_1)(x, y_1, y_2) + v(g_2)(x, y_2, y_1) \\ &= f_1(x, y_1, y_2) + f_2(x, y_2, y_1) \end{aligned} \tag{6}$$

be a Bergman type $*$-identity in symmetric variables. We consider its consequence (in symmetric variables)

$$\begin{aligned} A &= \alpha f(y_1 = xy_1 + y_1 x) + \beta f(y_2 = xy_2 + y_2 x) + \gamma(xf + fx) \\ &= \alpha[f(y_1 = xy_1) + f(y_1 = y_1 x)] \\ &+ \beta[f(y_2 = xy_2) + f(y_2 = y_2 x)] + \gamma(xf + fx) \\ &= \alpha x f_1 + \alpha f_2(y_1 = xy_1) + \alpha f_1(y_1 = y_1 x) + \alpha f_2 x \\ &+ \beta f_1(y_1 = y_1 x) + \beta x f_2 + \beta f_1 x + \beta f_2(y_1 = xy_1) \\ &+ \gamma x f_1 + \gamma x f_2 + \gamma f_1 x + \gamma f_2 x = A_1(x, y_1, y_2) + A_2(x, y_2, y_1). \end{aligned}$$

The commutative polynomials corresponding to the parts $A_1(x, y_1, y_2)$ and $A_2(x, y_2, y_1)$ are respectively

$$\begin{aligned} g_{11} = g_{(1,2)} &= [(\alpha+\gamma)t_1 + (\alpha+\beta)t_2 + (\beta+\gamma)t_3]g_1, \\ g_{21} = g_{(2,1)} &= [(\beta+\gamma)t_1 + (\alpha+\beta)t_2 + (\alpha+\gamma)t_3]g_2. \end{aligned} \tag{7}$$

1. For $\alpha = 1, \beta = \gamma = 0$ (7) gives

$$g_{11} = g_1^{'} = (t_1 + t_2)g_1,$$
$$g_{21} = g_2^{'} = (t_2 + t_3)g_2.$$

2. For $\alpha = \gamma = 0,\ \beta = 1$ we get

$$g_{11} = g_1^{''} = (t_2 + t_3)g_1,$$
$$g_{21} = g_2^{''} = (t_1 + t_2)g_2.$$

The linearization in y of the pointed in [5, Part 3] identity $[[x,y]^2,x] = 0$ in symmetric variables is a Bergman type identity of type (3) for which (following (6))

$$g_{(1,2)} = g_{(2,1)} = g_0 = (t_1 - t_2)(t_1 - t_3)(t_2 - t_3). \tag{8}$$

The linearization in y of the identity $[[x^2,y]^2,x] = 0$ (in skew-symmetric variables) is a Bergman type identity of type (3). In this case (7) gives

$$\begin{aligned} g_{(1,2)} = g_{(2,1)} = g &= (t_1^2 - t_2^2)(t_1 - t_3)(t_2^2 - t_3^2) \\ &= (t_1 + t_2)(t_2 + t_3)g_0. \end{aligned} \tag{9}$$

We want to show that the linearization of $[[x^2,y]^2,x] = 0$, which corresponds to (9), is a consequence in symmetric variables of the linearization of $[[x,y]^2,x] = 0$, corresponding to (8).

Applying the first case 1 to the identity in symmetric variables

$$f(x, y_1, y_2) = v(g_0)(x, y_1, y_2) + v(g_0)(x, y_2, y_1),$$

we obtain the identity $f^{'}(x, y_1, y_2)$ for which $g_1^{'} = (t_1 + t_2)g_0$ and $g_2^{'} = (t_2 + t_3)g_0$.

Now we apply the second step to $f^{'}(x, y_1, y_2)$ and get $f^{''}(x, y_1, y_2)$ for which $g_1^{''} = (t_2 + t_3)(t_1 + t_2)g_0$ and $g_2^{''} = (t_1 + t_2)(t_2 + t_3)g_0$. Thus $f^{''}(x, y_1, y_2)$ is a consequence of $f(x, y_1, y_2)$ in symmetric variables and holds for $M_4(K,*)$ since $g_1^{''} = g_2^{''} = g$ is as in (9).

We point that the identity in skew-symmetric variables of degree 7 [3, Theorem 5] is not a $*$-identity in symmetric variables. Hence the Bergman type of the considered polynomials is essential.

Theorem 2. *All Bergman type polynomials f of degree 7, which are $*$-identities both in symmetric and skew-symmetric variables, are of the form kf_0, $k \in K$, for $f_0(x, y_1, y_2) = f_1 + f_2 = v(g)(x, y_1, y_2) + v(g)(x, y_2, y_1)$, where $g = (t_1^2 - t_2^2)(t_1 - t_3)(t_2^2 - t_3^2)$.*

Proof. According to Theorem 1, Proposition 1 and the notations in Theorem 2, we write f as $f = \alpha f_1 + \beta f_2$ for $\alpha, \beta \in K$. As $f(y_1 \leftrightarrow y_2) = 0$ is an identity too, we get $\alpha f_2 + \beta f_1 = 0$. It means that $(\alpha - \beta)f_1 + (\beta - \alpha)f_2 = 0$. If $\alpha - \beta \neq 0$, then $f_1 - f_2 = 0$ is an identity for $M_4(K,*)$. The identity given before Proposition

2 leads to the identity $f_1 + f_2 = 0$. Thus $f_1 = 0$ and $f_2 = 0$ are identities in skew-symmetric variables. Calculating $f_i(\rho_1(e_{11}-e_{33})+\rho_2(e_{22}-e_{44}), e_{12}-e_{43}, e_{14}+e_{23})$ for $i = 1, 2$, we get

$$\begin{aligned} f_1 &= 2\rho_1(\rho_1^2-\rho_2^2)^2 e_{13} \neq 0, \\ f_2 &= -2\rho_1(\rho_1^2-\rho_2^2)^2 e_{13} \neq 0, \end{aligned}$$

a contradiction. Thus $\alpha - \beta = 0$ and $f = kf_0$.

Now we continue the investigations in $M_6(K,*)$.

For $n = 3$ the commutative polynomial in Proposition 1 is denoted by g_0, and $v(g_0)(x, y_{i_1}, y_{i_2}, y_{i_3})$ – by $f_0(x, y_{i_1}, y_{i_2}, y_{i_3})$.

Proposition 3. [4, Theorem 3] *The polynomial*

$$P(x, y_1, y_2, y_3) = f_0(x, y_{i_1}, y_{i_2}, y_{i_3}) + f_0(x, y_{i_3}, y_{i_2}, y_{i_1})$$

for all (i_1, i_2, i_3) *is a* $*$*-identity of degree* 14 *in skew-symmetric variables for* $M_6(K,*)$.

Using the notations

$f_1 = v(g_0)(x, y_1, y_2, y_3)$, $f_2 = v(g_0)(x, y_1, y_3, y_2)$,

$f_3 = v(g_0)(x, y_2, y_1, y_3)$, $f_4 = v(g_0)(x, y_3, y_2, y_1)$,

$f_5 = v(g_0)(x, y_2, y_3, y_1)$ and $f_6 = v(g_0)(x, y_3, y_1, y_2)$,

we get that

$$\alpha(f_1 + f_4) + \beta(f_2 + f_5) + \gamma(f_3 + f_6) = 0, \ \ \alpha, \beta, \gamma \in K, \tag{10}$$

is a $*$-identity in skew-symmetric variables.

We describe the class of all Bergman type identities of degree 14 in skew-symmetric variables for $M_6(K,*)$.

Theorem 3. *All Bergman type* $*$*-identities of degree* 14 *in skew-symmetric variables for* $M_6(K,*)$ *have the form* (10).

Proof. According to Proposition 1, any identity of the considered type has the form $\sum_{i=1}^{6} \alpha_i f_i$. As $\alpha_1(f_1 + f_4) = 0$, $\alpha_2(f_2 + f_5) = 0$ and $\alpha_3(f_3 + f_6) = 0$ are identities in skew-symmetric variables, we get the identity

$$f = (\alpha_4 - \alpha_1)f_4 + (\alpha_5 - \alpha_2)f_5 + (\alpha_6 - \alpha_3)f_6 = 0.$$

Considering $f(y_1 \leftrightarrow y_2) = 0$, $f(y_2 \leftrightarrow y_3) = 0$ and $f(y_3 \leftrightarrow y_1) = 0$, we get the following system:

$$\begin{aligned} (\alpha_4 - \alpha_1)f_4 + (\alpha_5 - \alpha_2)f_5 + (\alpha_6 - \alpha_3)f_6 = 0, \\ (\alpha_4 - \alpha_1)f_6 + (\alpha_5 - \alpha_2)f_2 + (\alpha_6 - \alpha_3)f_4 = 0, \\ (\alpha_4 - \alpha_1)f_5 + (\alpha_5 - \alpha_2)f_4 + (\alpha_6 - \alpha_3)f_3 = 0, \\ (\alpha_4 - \alpha_1)f_1 + (\alpha_5 - \alpha_2)f_3 + (\alpha_6 - \alpha_3)f_2 = 0. \end{aligned}$$

Using the notations $a = (\alpha_4 - \alpha_1)$, $b = (\alpha_5 - \alpha_2)$ and $c = (\alpha_6 - \alpha_3)$ and Proposition 3, the matrix A of the considered homogeneous system for the unknowns f_4, f_5 and f_6 has the form

$$\begin{pmatrix} a & b & c \\ c & -b & a \\ b & a & -c \\ -a & -c & -b \end{pmatrix}.$$

Elementary transformations on the matrix and *Mathematica* calculations prove that special polynomials are not $*$-identities in skew-symmetric variables and they lead to the only trivial solution for a, b, c. Thus $\alpha_1 = \alpha_4$, $\alpha_2 = \alpha_5$ and $\alpha_3 = \alpha_6$, and we get (10).

The main Theorem 1 applied for $M_6(K,*)$ gives

Proposition 4. *All Bergman type $*$-identities of degree* 14 *both in symmetric and skew-symmetric variables for $M_6(K,*)$ have the form* (10).

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