ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

COMPUTATION OF THE VERTEX FOLKMAN NUMBERS $F(2, 2, 2, 3; 5)$ AND $F(2, 3, 3; 5)$

NEDYALKO NENOV

In this note we show that the exact value of the vertex Folkman numbers $F(2, 2, 2, 3; 5)$ and $F(2, 3, 3; 5)$ is 12.

Keywords: vertex Folkman graph, vertex Folkman number

2000 MSC: 05C55

1. NOTATIONS

We consider only finite, non-oriented graphs, without loops and multiple edges. The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We call a p-clique of G a set of p vertices, each two of which are adjacent. The biggest natural number p such that the graph G contains a p-clique is denoted by $\mathrm{cl}(G)$ (the clique number of G).

If $W \subseteq V(G)$, then: $G[W]$ is the subgraph of G induced by W and $G - W$ is the subgraph of G induced by $V(G) \setminus W$. We shall use also the following notations:

$\overline{G}$ — the complement of the graph G;

$\alpha(G)$ — the independence number of G;

$N_G(v)$, $v \in V(G)$ — the set of all vertices of G adjacent to v;

K_n — the complete graph of n vertices;

C_n — the simple cycle of n vertices;

$\chi(G)$ — the chromatic number of G.

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y],\ x \in V(G_1),\ y \in V(G_2)\}$.

The Ramsey number $R(p,q)$ is the smallest natural n such that for an arbitrary n-vertex graph G either $\alpha(G) \geq p$ or $\operatorname{cl}(G) \geq q$. We need the equality $R(3,3) = 6$, [3].

2. VERTEX FOLKMAN NUMBERS AND THE MAIN RESULT

Definition 2.1. Let G be a graph, $a_1, \ldots, a_r$ be positive integers and let

$$V(G) = V_1 \cup \ldots \cup V_r, \quad V_i \cap V_j = \emptyset, \ i \neq j,$$

be an r-coloring of the vertices of G. This coloring is said to be $(a_1, \ldots, a_r)$-free if for all $i \in \{1, \ldots, r\}$ the graph G does not contain a monochromatic a_i-clique of color i. The symbol $G \to (a_1, \ldots, a_r)$ means that every r-coloring of $V(G)$ is not $(a_1, \ldots, a_r)$-free.

The graph G such that $G \to (a_1, \ldots, a_r)$ is called a vertex Folkman graph. We put

$$F(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \to (a_1, \ldots, a_r) \text{ and } \operatorname{cl}(G) < q\}.$$

It is clear that from $G \to (a_1, \ldots, a_r)$ it follows that $\operatorname{cl}(G) \geq \max\{a_1, \ldots, a_r\}$. Folkman, [2], proves that there exists a graph G such that $G \to (a_1, \ldots, a_r)$ and $\operatorname{cl}(G) = \max\{a_1, \ldots, a_r\}$. Therefore, if $q > \max\{a_1, \ldots, a_r\}$, then the numbers $F(a_1, \ldots, a_r; q)$ exist. Those numbers are called vertex Folkman numbers.

Let $a_1, \ldots, a_r$ be positive integers. We put

$$m = \sum_{i=1}^{r}(a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, \ldots, a_r\}. \tag{1}$$

Obviously, $K_m \to (a_1, \ldots, a_r)$ and $K_{m-1} \not\to (a_1, \ldots, a_r)$. Hence, if $q \geq m+1$, then $F(a_1, \ldots, a_r; q) = m$. The numbers $F(a_1, \ldots, a_r; m)$ exist only if $m \geq p+1$. For those numbers the following is known:

Theorem A ([4]). *Let $a_1, \ldots, a_r$ be positive integers and let m and p satisfy (1), where $m \geq p+1$. Then $F(a_1, \ldots, a_r; m) = m + p$. If $G \to (a_1, \ldots, a_r)$, $\operatorname{cl}(G) < m$ and $|V(G)| = m + p$, then $G = K_{m-p-1} + \overline{C}_{2p+1}$.*

Remark. The proof of Theorem A, given in [4], is based on [4, Lemma 1, p. 251]. But the proof of this lemma is not correct, because the sentence "If we delete both endpoints of any its edges adjacent to $\{x, y\}$, then $\alpha(G)$ decreases again." is not true (see p.252).

A correct proof of Theorem A is given in [13] (see also p.66, Theorem 7.4 in this volume).

The numbers $F(a_1, \ldots, a_r; m-1)$ exist only if $m \geq p+2$. For those numbers the following is known:

Theorem B ([13]). *Let $a_1, \ldots, a_r$ be positive integers. Let m and p satisfy (1), where $m \geq p+2$. Then $F(a_1, \ldots, a_r; m-1) \geq m+p+2$.*

Theorem C ([14]). *Let $a_1, \dots, a_r$ be positive integers and let m and p satisfy* (1). *Let $m \ge p+2$, $G \to (a_1, \dots, a_r)$ and $\mathrm{cl}(G) < m-1$. Then:*

(a) $|V(G)| \ge m + p + \alpha(G) - 1$;

(b) *if* $|V(G)| = m + p + \alpha(G) - 1$, *then* $|V(G)| \ge m + 3p$.

It is clear that for each permutation φ of the symmetric group S_r

$$G \to (a_1, \dots, a_r) \iff G \to (a_{\varphi(1)}, \dots, a_{\varphi(r)}).$$

Note that if $a_1 = 1$, then $F(a_1, \dots, a_r; q) = F(a_2, \dots, a_r; q)$. Therefore, we can assume that $2 \le a_1 \le \dots \le a_r$.

The next theorem implies that, in the special situation $a_1 = \dots = a_r = 2$, $r \ge 5$, the inequality from Theorem B is exact.

Theorem D.

$$F(\underbrace{2, \dots, 2}_{r}; r) = \begin{cases} 11, & r = 3 \text{ or } r = 4; \\ r+5, & r \ge 5. \end{cases}$$

It is clear that $G \to (\underbrace{2, \dots, 2}_{r}) \iff \chi(G) \ge r+1$.

Mycielski in [5] presents an 11-vertex graph G such that $G \to (2,2,2)$ and $\mathrm{cl}(G) = 2$, proving that $F(2,2,2;3) \le 11$. Chvátal, [1], proves that Mycielski graph is the smallest such graph and hence $F(2,2,2;3) = 11$. The inequality $F(2,2,2,2;4) \ge 11$ is proved in [8] and inequality $F(2,2,2,2;4) \le 11$ is proved in [7] and [12] (see also [9]). The equality $F(\underbrace{2, \dots, 2}_{r}; r) = r+5$, $r \ge 5$, is proved in [7], [12] and later in [4]. Only few other numbers of the type $F(a_1, \dots, a_r; m-1)$ are known, namely: $F(3,3;4) = 14$ (the inequality $F(3,3;4) \le 14$ is proved in [6] and the opposite inequality $F(3,3;4) \ge 14$ is verified by means of computers in [18]); $F(3,4;5) = 13$, [10]; $F(2,2,4;5) = 13$, [11]; $F(4,4;6) = 14$, [15]; $F(2,2,2,4;6) = F(2,3,4;6) = 14$, [16].

In this paper we will calculate another two numbers of this type.

Theorem 2.1. $F(2,2,2,3;5) = F(2,3,3;5) = 12$.

3. THE LEMMAS

We consider the graph P, whose complementary graph $\overline{P}$ is given in Fig. 1. For this graph we put

$$A = \{a_1, \dots, a_8\}, \quad B = \{b_1, b_2, b_3, b_4\}.$$

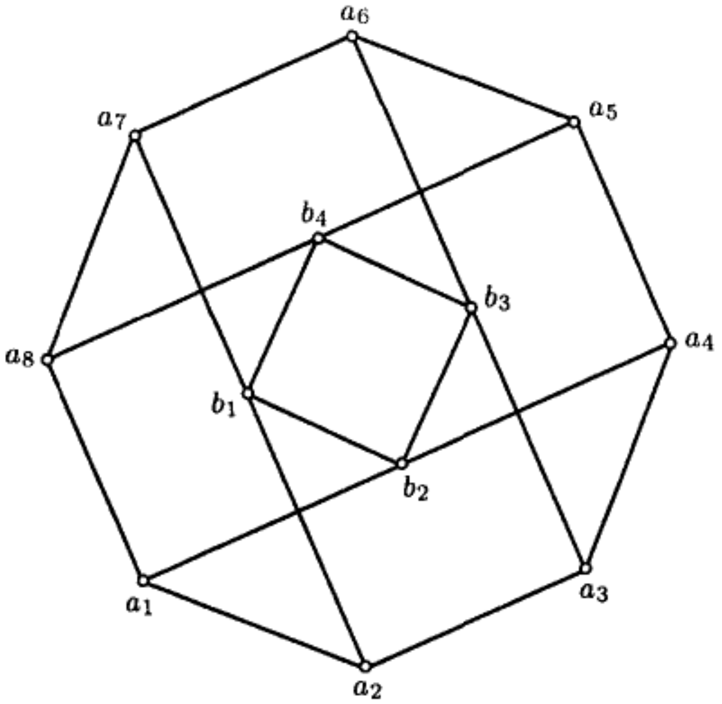

Fig. 1. Graph $\overline{P}$

Lemma 3.1 (Main Lemma). $P \to (2,3,3)$.

To prove the main Lemma, we make use of the next lemmas.

Lemma 3.2. *Let* $W \subseteq V(\overline{P})$ *and* $\overline{P}[W] = C_5$.
(a) *If* $W \cap B = \{b_1\}$, *then* $W = \{b_1, a_1, a_2, a_7, a_8\}$.
(b) *If* $W \cap B = \{b_2\}$, *then* $W = \{b_2, a_1, a_2, a_3, a_4\}$.
(c) *If* $W \cap B = \{b_3\}$, *then* $W = \{b_3, a_3, a_4, a_5, a_6\}$.
(d) *If* $W \cap B = \{b_4\}$, *then* $W = \{b_4, a_5, a_6, a_7, a_8\}$.

Proof. It is sufficient to prove the proposition (a).

Let $W \cap B = \{b_1\}$. From $b_2, b_4 \notin W$ and $\overline{P}[W] = C_5$ it follows that $a_2, a_7 \in W$. From $a_7 \in W$ it follows that $a_8 \in W$ or $a_6 \in W$. From $a_2 \in W$ it follows that $a_1 \in W$ or $a_3 \in W$. Since in $\{a_1, a_3, a_6, a_8\}$ only a_1 and a_8 are adjacent in $\overline{P}$, we have $W = \{b_1, a_1, a_2, a_7, a_8\}$. □

Lemma 3.3. *Let* $W \subseteq V(\overline{P})$, $\overline{P}[W] = C_5$ *and* $|W \cap B| = 2$. *Then the two vertices of* $W \cap B$ *are adjacent in* $\overline{P}$.

Proof. Assume the contrary and let for example $W = \{b_1, b_3\}$. From $\overline{P}[W] = C_5$ it follows that there exists $u \in W$ such that $u \in N_{\overline{P}}(b_1) \cap N_{\overline{P}}(b_3)$. Since $N_{\overline{P}}(b_1) \cap N_{\overline{P}}(b_3) = \{b_2, b_4\}$, this contradicts equality $W = \{b_1, b_3\}$. □

Lemma 3.4. *Let* $W \subseteq V(\overline{P})$ *and* $\overline{P}[W] = C_5$.
(a) *If* $W \cap B = \{b_1, b_2\}$, *then* $W = \{b_1, b_2, a_1, a_7, a_8\}$ *or* $W = \{b_1, b_2, a_2, a_3, a_4\}$.
(b) *If* $W \cap B = \{b_2, b_3\}$, *then* $W = \{b_2, b_3, a_1, a_2, a_3\}$ *or* $W = \{b_2, b_3, a_4, a_5, a_6\}$.
(c) *If* $W \cap B = \{b_3, b_4\}$, *then* $W = \{b_3, b_4, a_3, a_4, a_5\}$ *or* $W = \{b_3, b_4, a_6, a_7, a_8\}$.

(d) *If* $W \cap B = \{b_1, b_4\}$, *then* $W = \{b_1, b_4, a_5, a_6, a_7\}$ *or* $W = \{b_1, b_4, a_1, a_2, a_8\}$.

Proof. It is sufficient to prove the proposition (a).

Let $W \cap B = \{b_1, b_2\}$. From $b_1 \in W$ and $b_4 \notin W$ it follows that $a_2 \in W$ or $a_7 \in W$. Let $a_2 \in W$. Since $\overline{P}[a_1, a_2, b_1, b_2] = C_4$, we have $a_1 \notin W$. Hence, $a_3 \in W$. Therefore, from $\overline{P}[W] = C_5$ it follows that $W = \{b_1, b_2, a_2, a_3, a_4\}$. Let $a_7 \in W$. From $\overline{P}[W] = C_5$ it follows that $a_6 \in W$ or $a_8 \in W$. Since $N_{\overline{P}}(a_6) \cap N_{\overline{P}}(b_2) = \{b_3\}$ and $b_3 \notin W$, we have $a_8 \in W$. From $N_{\overline{P}}(a_8) \cap N_{\overline{P}}(b_2) = \{a_1\}$ it follows that $W = \{b_1, b_2, a_1, a_7, a_8\}$. □

Lemma 3.5. *Let* $W \subseteq V(\overline{P})$ *and* $\overline{P}[W] = C_5$.

(a) *If* $W \cap B = \{b_1, b_2, b_3\}$, *then*
$$W = \{b_1, b_2, b_3, a_2, a_3\} \quad \textit{or} \quad W = \{b_1, b_2, b_3, a_6, a_7\}.$$
(b) *If* $W \cap B = \{b_2, b_3, b_4\}$, *then*
$$W = \{b_2, b_3, b_4, a_4, a_5\} \quad \textit{or} \quad W = \{b_2, b_3, b_4, a_1, a_8\}.$$
(c) *If* $W \cap B = \{b_1, b_3, b_4\}$, *then*
$$W = \{b_1, b_3, b_4, a_6, a_7\} \quad \textit{or} \quad W = \{b_1, b_3, b_4, a_2, a_3\}.$$
(d) *If* $W \cap B = \{b_1, b_2, b_4\}$, *then*
$$W = \{b_1, b_2, b_4, a_1, a_8\} \quad \textit{or} \quad W = \{b_1, b_2, b_4, a_4, a_5\}.$$

Proof. It is sufficient to prove the proposition (a). Let $W \cap B = \{b_1, b_2, b_3\}$. From $b_1 \in W$ and $\overline{P}[W] = C_5$ it follows that $a_2 \in W$ or $a_7 \in W$. Let $a_2 \in W$. Since $N_{\overline{P}}(a_2) \cap N_{\overline{P}}(b_3) = \{a_3\}$, we have $W = \{b_1, b_2, b_3, a_2, a_3\}$. If $a_7 \in W$, then from $N_{\overline{P}}(a_7) \cap N_{\overline{P}}(b_3) = \{a_6\}$ it follows that $W = \{b_1, b_2, b_3, a_6, a_7\}$. □

4. A PROOF OF THE MAIN LEMMA

Assume that $P \nrightarrow (2,3,3)$ and let $V_1 \cup V_2 \cup V_3$ be a $(2,3,3)$-free 3-coloring of $V(P)$. From $\alpha(P) = 2$ it follows that

$$|V_1| \leq 2. \tag{2}$$

Since V_i, $i = 2, 3$, contains no 3-clique, from $\alpha(P) = 2$ and $R(3,3) = 6$ it follows that

$$|V_i| \leq 5, \quad i = 2, 3. \tag{3}$$

The equality $|V(P)| = 12$ together with (2) and (3) imply that $|V_1| = 2$, $|V_2| = |V_3| = 5$. We put $G_i = \overline{P}[V_i]$, $i = 2, 3$. Since $\alpha(G_i) = \mathrm{cl}(G_i) = 2$, from $|V_i| = 5$, $i = 2, 3$, it follows that $G_2 = G_3 = C_5$. Obviously, $\overline{P}[A] = C_8$. Hence $V_i \cap B \neq \emptyset$, $i = 2, 3$. Assume that $|V_2 \cap B| \leq |V_3 \cap B|$. From $|B| = 4$ it follows that $1 \leq |V_2 \cap B| \leq 2$.

Case 1. $|V_2 \cap B| = 1$. Without a loss of generality we can assume that $V_2 \cap B = \{b_1\}$. According to Lemma 3.2(a), $V_2 = \{b_1, a_1, a_2, a_7, a_8\}$.

Subcase 1a. $|V_3 \cap B| = 1$. Suppose that $V_3 \cap B = \{b_2\}$ or $V_3 \cap B = \{b_4\}$. Then, according to Lemma 3.2, $V_2 \cap V_3 \neq \emptyset$, which is a contradiction. Let $V_3 \cap B = \{b_3\}$.

Then $V_3 = \{b_3, a_3, a_4, a_5, a_6\}$ (see Lemma 3.2(c)). Hence $V_1 = \{b_2, b_4\}$. This contradicts the assumption that V_1 is independent in P.

Subcase 1b. $|V_3 \cap B| = 2$. According to Lemma 3.3, $V_3 \cap B = \{b_2, b_3\}$ or $V_3 \cap B = \{b_3, b_4\}$. Without a loss of generality we can assume that $V_3 \cap B = \{b_2, b_3\}$. From $V_2 \cap V_3 = \emptyset$ and Lemma 3.4(b) it follows that $V_3 = \{b_2, b_3, a_4, a_5, a_6\}$. Hence $V_1 = \{a_3, b_4\}$. This contradicts the assumption that V_1 is independent in P.

Subcase 1c. $|V_3 \cap B| = 3$. It is clear that $V_3 \cap B = \{b_2, b_3, b_4\}$. From $V_2 \cap V_3 = \emptyset$ and Lemma 3.5(b) it follows that $V_3 = \{b_2, b_3, b_4, a_4, a_5\}$. Hence $V_1 = \{a_3, a_6\}$. This contradicts the assumption that V_1 is an independent set in P.

Case 2. $|V_2 \cap B| = 2$. It is clear that $|V_3 \cap B| = 2$. According to Lemma 3.3, we can assume that $V_2 \cap B = \{b_1, b_2\}$ and $V_3 \cap B = \{b_3, b_4\}$. Because of the Lemma 3.4(a) we have the following two subcases:

Subcase 2a. $V_2 = \{b_1, b_2, a_2, a_3, a_4\}$. From Lemma 3.4(c) and $V_2 \cap V_3 = \emptyset$ it follows that $V_3 = \{b_3, b_4, a_6, a_7, a_8\}$. Hence $V_1 = \{a_1, a_5\}$. This contradicts the assumption that V_1 is independent in P.

Subcase 2b. $V_2 = \{b_1, b_2, a_1, a_7, a_8\}$. From Lemma 3.4(c) and $V_2 \cap V_3 = \emptyset$ it follows that $V_3 = \{b_3, b_4, a_3, a_4, a_5\}$. Hence $V_2 = \{a_2, a_6\}$. This contradicts the assumption that V_1 is independent in P.

5. A PROOF OF THEOREM 2.1

It is obvious that from $G \to (2,3,3)$ it follows that $G \to (2,2,2,3)$. Therefore

$$F(2,2,2,3;5) \leq F(2,3,3;5).$$

From the above inequality it becomes clear that it is sufficient to prove that $F(2,3,3;5) \leq 12$ and $F(2,2,2,3;5) \geq 12$.

1. Proof of the inequality $F(2,3,3;5) \leq 12$. According to the main Lemma, $P \to (2,3,3)$. Since $\mathrm{cl}(P) = 4$ and $|V(P)| = 12$, we have $F(2,3,3;5) \leq 12$.

2. Proof of the inequality $F(2,2,2,3;5) \geq 12$. According to Theorem B, $F(2,2,2,3;5) \geq 11$. Assume that $F(2,2,2,3;5) = 11$ and let G be a graph such that $|V(G)| = 11$, $\mathrm{cl}(G) < 5$ and $G \to (2,2,2,3)$. From Theorem C(a) it follows that $\alpha(G) \leq 3$. According to Theorem C(b), $\alpha(G) \neq 3$. Hence

$$\alpha(G) = 2. \tag{4}$$

Assume that there exist $u, v \in V(G)$ such that $N_G(u) \supseteq N_G(v)$. It is clear that $\{u, v\} \notin E(G)$. From $F(2,2,2,3;5) \geq 11$ it follows that $G - v \not\to (2,2,2,3)$. Consider an arbitrary $(2,2,2,3)$-free 4-coloring of $G - v$. If we color the vertex v with the same color as the vertex u, we will obtain $(2,2,2,3)$-free 4-coloring of G, which is a contradiction. Therefore:

$$N_G(v) \not\subseteq N_G(u) \quad \text{for all } u, v \in V(G). \tag{5}$$

If $|N_G(v)| = |V(G)| - 1$ for some $v \in V(G)$, then $\mathrm{cl}(G - v) < 4$ and $G - v \to (2,2,2,2)$. This contradicts Theorem D. Hence, $|N_G(v)| \neq |V(G)| - 1$, $\forall v \in V(G)$. This, together with (5) imply that

$$|N_G(v)| \leq |V(G)| - 3 \quad \text{for all } v \in V(G). \tag{6}$$

From $F(2,2,4;5) = 13$, [11], it follows that $G \not\to (2,2,4)$. Let $V_1 \cup V_2 \cup V_3$ be $(2,2,4)$-free 3-coloring of $V(G)$. It follows from (4) that $|V_1| \leq 2$, $|V_2| \leq 2$. According to (6) and (4), we may assume that $|V_1| = |V_2| = 2$. We put $G_1 = G[V_3]$. It is clear that from $G \to (2,2,2,3)$ it follows that $G_1 \to (2,3)$. According to Theorem A, $G_1 = \overline{C}_7$ (Fig. 2). Let $V_1 = \{a,b\}$, $V_2 = \{c,d\}$ and $G_2 = G[a,b,c,d]$. From (4) it follows that $E(G_2)$ contains two independent edges. Without a loss of generality we can assume that $[a,c]$, $[b,d] \in E(G_2)$. It is sufficient to consider the next two cases.

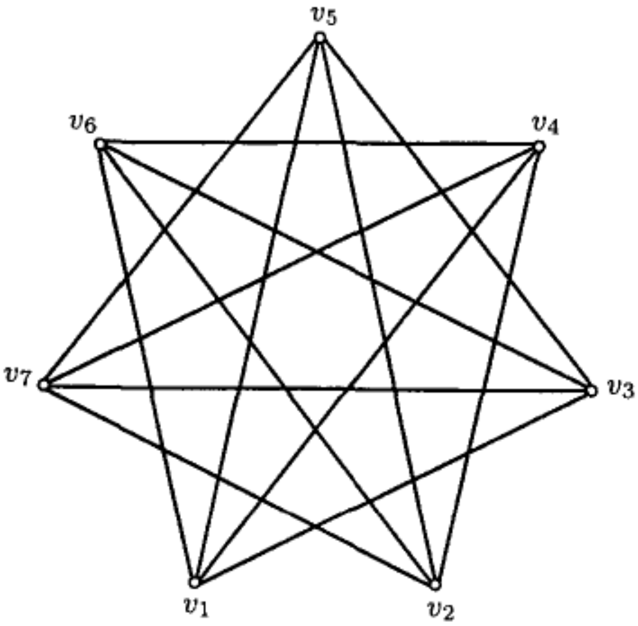

Fig. 2. Graph $\overline{C}_7$

Case 1. $E(G_2) = \{[a,c],\ [b,d]\}$. From $\mathrm{cl}(G) < 5$ it follows that one of the vertices a, c is not adjacent to some of the vertices $v_1, \ldots, v_7$ (see Fig. 2). Without a loss of generality we may assume that v_1 and a are not adjacent. Consider the 4-coloring

$$\{v_4, v_5\} \cup \{v_6, v_7\} \cup \{c, d\} \cup \{v_1, v_2, v_3, a, b\}.$$

Since $G \to (2,2,2,3)$, we have that $\{v_1, v_2, v_3, a, b\}$ contains a 3-clique. Hence $v_1, v_3 \in N_G(b)$. Similarly, $v_1, v_6 \in N_G(b)$. So, $v_1, v_3, v_6 \in N_G(b)$. Similarly, $v_1, v_3, v_6 \in N_G(d)$. Hence $\{v_1, v_3, v_6, b, d\}$ is a 5-clique, which is a contradiction.

Case 2. $E(G_2) \supseteq \{[a,c],\ [b,d],\ [a,d]\}$. As in case 1, we may assume that a and v_1 are not adjacent. Then from (4) it follows that $v_2, v_7 \in N_G(a)$. From (4) it follows also that a is adjacent to some of the vertices v_4, v_5. Without a loss of generality we may assume that v_4 and a are adjacent. So,

$$v_2, v_4, v_7 \in N_G(a). \tag{7}$$

From (7) and $\mathrm{cl}(G) < 5$ it follows that d is not adjacent to any of the vertices v_2, v_4, v_7. Hence, it is sufficient to consider the next three subcases.

Subcase 2a. The vertex d is not adjacent to v_2. Consider the 4-coloring

$$\{v_5, v_6\} \cup \{v_1, v_7\} \cup \{a, b\} \cup \{v_2, v_3, v_4, c, d\} \tag{8}$$

of $V(G)$. From $G \to (2,2,2,3)$ it follows that $\{v_2, v_3, v_4, c, d\}$ contains a 3-clique. Hence, $v_2, v_4 \in N_G(c)$. Similarly, $v_2, v_7 \in N_G(c)$. From (7) it follows that $\{v_2, v_4, v_7, a, c\}$ is a 5-clique, which contradicts $\mathrm{cl}(G) < 5$.

Subcase 2b. The vertex d is not adjacent to v_4. Consider the 4-coloring (8). As in the subcase 2a it follows that $v_2, v_4 \in N_G(c)$. Similarly, from the 4-coloring

$$\{v_1, v_7\} \cup \{v_2, v_3\} \cup \{a, b\} \cup \{v_4, v_5, v_6, c, d\}$$

it follows that $v_4, v_6 \in N_G(c)$. So,

$$v_2, v_4, v_6 \in N_G(c). \tag{9}$$

According to (7), (9) and $\mathrm{cl}(G) < 5$, the vertex c is not adjacent to v_7. Consider the 4-coloring

$$\{v_3, v_4\} \cup \{v_5, v_6\} \cup \{a, b\} \cup \{v_1, v_2, v_7, c, d\}.$$

Since $G \to (2,2,2,3)$, then $\{v_1, v_2, v_7, c, d\}$ contains a 3-clique. Hence, $v_2, v_7 \in N_G(d)$. Similarly, from $G \to (2,2,2,3)$ and the 4-coloring

$$\{v_1, v_2\} \cup \{v_3, v_4\} \cup \{a, b\} \cup \{v_5, v_6, v_7, c, d\}$$

it follows that $v_5, v_7 \in N_G(d)$. Then

$$v_2, v_5, v_7 \in N_G(d). \tag{10}$$

From (7), (9) and $\mathrm{cl}(G) < 5$ it follows that a and v_6 are not adjacent. From (7), (10) and $\mathrm{cl}(G) < 5$ it follows that a and v_5 are not adjacent. So, the vertex a is not adjacent to v_5 and v_6, which contradicts (4).

Subcase 2c. The vertex d is not adjacent to v_7. This subcase is analogous with subcase 2b.

6. THE EXTREMAL GRAPHS

By $G - e$, $e \in E(G)$, we denote the subgraph of G such that $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$.

Consider the graph $\overline{P}$ from Fig. 1. For this graph we set: $P_0 = P$, $P_1 = P - [a_1, a_6]$, $P_2 = P - [a_1, a_5]$, $P_3 = P_1 - [a_2, a_5]$, $P_4 = P_1 - [a_4, a_7]$, $P_5 = P_1 - [a_3, a_7]$, $P_6 = P_2 - [a_4, a_8]$, $P_7 = P_2 - [a_3, a_7]$, $P_8 = P_3 - [a_4, a_7]$, $P_9 = P_7 - [a_2, a_6]$, $P_{10} = P_8 - [a_3, a_8]$, $P_{11} = P_9 - [a_4, a_8]$.

We need the next theorem.

Theorem E, [17]. *Let the graph G be such that $|V(G)| = 12$, $\mathrm{cl}(G) = 4$ and $\alpha(G) = 2$. Then G is isomorphic to one of the graphs P_i, $i = 0, \dots, 11$.*

Definition 6.1. We say that the graph G is extremal if $|V(G)| = 12$, $\mathrm{cl}(G) < 5$, $G \to (2, 3, 3)$ or $G \to (2, 2, 2, 3)$.

According to Theorem C(a), if G is extremal, then $\alpha(G) \leq 4$. From Theorem C(b) it follows that $\alpha(G) \neq 4$. Hence $\alpha(G) = 2$ or $\alpha(G) = 3$. In this section we describe all critical graphs G with $\alpha(G) = 2$.

Theorem 6.1. *Let G be extremal graph such that $G \to (2, 3, 3)$ and $\alpha(G) = 2$. Then G is isomorphic to the graph P.*

Proof. According to Theorem E, the graph G is isomorphic to one of the graphs P_i, $i = 0, \dots, 11$. The 3-coloring

$$\{a_7, a_8\} \cup \{b_2, a_1, a_4, a_5, a_6\} \cup \{b_1, b_3, b_4, a_2, a_3\}$$

of P_1 is $(2, 3, 3)$-free and the 3-coloring

$$\{a_1, a_5\} \cup \{b_1, b_2, a_2, a_3, a_4\} \cup \{b_3, b_4, a_6, a_7, a_8\}$$

of P_2 is $(2, 3, 3)$-free. Hence G is not a subgraph of P_1 and P_2. Thus $G = P$. □

Theorem 6.2. $P_i \to (2, 2, 2, 3)$, $i = 0, \dots, 11$. *If an extremal graph G is such that $G \to (2, 2, 2, 3)$ and $\alpha(G) = 2$, then G is isomorphic to one of the graphs P_i, $i = 0, \dots, 11$.*

Proof. Let $V_1 \cup V_2 \cup V_3 \cup V_4$ be a 4-coloring of $V(P_i)$ and V_i, $i = 1, 2, 3$, be independent. From $\alpha(P_i) = 2$ it follows that $|V_i| \leq 2$, $i = 1, 2, 3$. Hence $|V_4| \geq 6$. From $\alpha(P_i) = 2$ and $R(3, 3) = 6$ it follows that V_4 contains a 3-clique. Thus P_i does not have a $(2, 2, 2, 3)$-free 4-coloring and hence $P_i \to (2, 2, 2, 3)$. According to Theorem E, the graph G is isomorphic to one of the graphs P_i, $i = 0, \dots, 11$. □

7. THE VERTEX FOLKMAN NUMBERS $F(2, \dots, 2, p; q)$ AND THE RAMSEY NUMBERS $R(3, q)$

Theorem 7.1. *Let $p \geq 2$, r and q be positive integers such that*

$$R(3, p) + 2r < R(3, q). \tag{11}$$

Then $F(\underbrace{2, \dots, 2}_{r}, p; q) \leq R(3, p) + 2r$.

Proof. Let G be a graph such that $|V(G)| = R(3, p) + 2r$, $\mathrm{cl}(G) < q$ and

$$\alpha(G) = 2. \tag{12}$$

According to (11), the graph G exists. Let $V_1 \cup \ldots \cup V_{r+1}$ be an $(r+1)$-coloring of $V(G)$. Suppose that V_i, $i = 1, \ldots, r$, are independent. From (12) it follows that $|V_i| \leq 2$, $i = 1, \ldots, r$. Hence $|V_{r+1}| \geq R(3,p)$. According to the definition of $R(3,p)$ and (12), V_{r+1} contains a p-clique. Thus G does not have a $(\underbrace{2,\ldots,2}_{r},p)$-free coloring and hence $G \to (\underbrace{2,\ldots,2}_{r},p)$. From $\mathrm{cl}(G) < q$ and $|V(G)| = R(3,p) + 2r$ it follows that $F(\underbrace{2,\ldots,2}_{r},p;q) \leq R(3,p) + 2r$. □

Consider the table of the known Ramsey numbers $R(3,p)$, [19]:

p	3	4	5	6	7	8	9	10
$R(3,p)$	6	9	14	18	23	28	36	40–43

From this table and Theorem 7.1 it follows:

$F(2,2,4;5) \leq 13$ (in [11] it is proved $F(2,2,4;5) = 13$);
$F(2,2,6;7) \leq 22$ (in [11] it is proved $F(2,2,6;7) \leq 26$);
$F(2,2,7;8) \leq 27$ (in [11] it is proved $F(2,2,7;8) \leq 30$);
$F(2,2,8;9) \leq 32$ (in [11] it is proved $F(2,2,8;9) \leq 34$);
$F(2,2,9;10) \leq 40$ if $R(3,10) \neq 40$ (in [11] it is proved $F(2,2,9;10) \leq 38$);
$F(2,2,2,3;5) \leq 12$ (according to Theorem 2.1, $F(2,2,2,3;5) = 12$);
$F(2,2,2,5;7) \leq 20$ (in [11] it is proved $F(2,2,2,5;7) \leq 23$).

8. ON THE NUMBERS $F(\underbrace{2,\ldots,2}_{r},p;p+r-1)$

We put $F(\underbrace{2,\ldots,2}_{r},p;p+r-1) = F_r(2,p)$.

The proof of Theorem 5 from [13] establishes the following statement:

Theorem F. *Let $G \to (\underbrace{2,\ldots,2}_{s},p)$. Then $K_r + G \to (\underbrace{2,\ldots,2}_{r+s},p)$ for any r.*

From Theorem 2.1, Theorem F and Theorem B it follows that

$$r + 8 \leq F_r(2,3) \leq r + 9, \quad r \geq 3.$$

The exact value of $F_2(2,3) = F(2,2,3;4)$ is unknown.

From Theorem B, Theorem F and the inequalities $F_2(2,6) \leq 22$, $F_2(2,7) \leq 27$, $F_2(2,8) \leq 32$ and $F(2,2,2,5;7) \leq 20$ it follows that

$$r + 14 \leq F_r(2,6) \leq r + 20, \quad r \geq 2;$$
$$r + 16 \leq F_r(2,7) \leq r + 25, \quad r \geq 2;$$

$$r + 18 \leq F_r(2,8) \leq r + 30, \quad r \geq 2;$$
$$r + 12 \leq F_r(2,5) \leq r + 17, \quad r \geq 3.$$

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Received on October 17, 2001

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: nenov@fmi.uni-sofia.bg