THE TRIGONOMETRIC ANALOGUE OF THE TAYLOR'S FORMULA AND ITS APPLICATION

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A new approach to establishing generalized Taylor's expansions is used to prove the trigonometric analogue of the Taylor's formula. We derive point-wise estimates of the error in the trigonometric interpolation and approximation by convolutional linear operators.

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1. INTRODUCTION

We consider the function spaces $L_p^*[-\pi, \pi]$, $1 \leq p < \infty$, and $C^*[-\pi, \pi]$, where

$$L_p^*[-\pi, \pi] = \{f : \mathbf{R} \to \mathbf{R} : f(x + 2\pi) = f(x) \text{ a.e.}, f|_{[-\pi,\pi]} \in L_p[-\pi, \pi]\},$$
$$C^*[-\pi, \pi] = \{f \in C(\mathbf{R}) : f(x + 2\pi) = f(x)\},$$

normed, respectively, with the usual L_p-norm over the interval $[-\pi, \pi]$ for $1 \leq p < \infty$, denoted by $\| \cdot \|_p$, and the uniform norm over the interval $[-\pi, \pi]$, denoted by $\| \cdot \|_\infty$.

In a recent paper (see [1]) we have introduced a new modulus of smoothness, which describes the rate of the best trigonometric approximation. It is defined by

$$\omega_r^T(f; t)_p := \sup_{0 < h \leq t} \|\Delta_h^{2r-1} \mathcal{F}_{r-1} f\|_p, \ r = 1, 2, \ldots,$$

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where

$$\Delta_h^{2r-1} f(x) := \sum_{k=0}^{2r-1} (-1)^k \binom{2r-1}{k} f(x + ((2r-1)/2 - k)h)$$

is the symmetric finite difference of order $2r - 1$,

$$\mathcal{F}_{r-1}(f, x) = f(x) + \int_0^x \mathcal{K}_{r-1}(t) f(x - t)\, dt$$

and

$$\mathcal{K}_{r-1}(t) = \sum_{j=1}^{r-1} \frac{a_j^{(r-1)}}{(2j-1)!} t^{2j-1}, \quad a_j^{(r-1)} = \sum_{1 \leq l_1 < \cdots < l_j \leq r-1} (l_1 \cdots l_j)^2.$$

It is shown in [1] that for the rate of the best trigonometric approximation $E_n^T(f)_p := \inf_{\tau \in T_n} \|f - \tau\|_p$, T_n being the set of all trigonometric polynomials of degree at most n, we have

$$E_n^T(f)_p \leq C_r \omega_r^T(f; n^{-1})_p, \quad n \geq r - 1, \tag{1.1}$$

and

$$\omega_r^T(f; t)_p \leq C_r t^{2r-1} \sum_{r-1 \leq k \leq 1/t} (k+1)^{2r-2} E_k^T(f)_p, \quad 0 < t \leq \frac{1}{r}. \tag{1.2}$$

Moreover, we have $\omega_r^T(f; t)_p \equiv 0$ if and only if $f \in T_{r-1}$. In that sense the new modulus of smoothness describes the rate of the best trigonometric approximation more precisely than the classical one. The modulus of smoothness $\omega_r^T(f; t)_p$ possesses properties similar to those of the classical one, as it is shown in [1].

Let $L_n : L_p^*[-\pi, \pi] \to L_p^*[-\pi, \pi]$, $1 \leq p < \infty$, or $L_n : C^*[-\pi, \pi] \to C^*[-\pi, \pi]$, be a bounded linear operator that preserves the trigonometric polynomials of degree n. Then the well-known Lebesgue inequality

$$\|f - L_n f\|_p \leq (1 + \|L_n\|) E_n^T(f)_p$$

and the Jackson-type estimate (1.1) imply

$$\|f - L_n f\|_p \leq C_r (1 + \|L_n\|)\, \omega_r^T(f, n^{-1})_p, \quad n \geq r - 1.$$

Similar estimates, using the classical periodic modulus of smoothness, are known. For instance, G. P. Nevai has proved in [3] the following generalization of a result of S. M. Nikolskii:

$$\|f - t_n f\|_\infty \leq 2^{-r} \omega_r\left(f; \frac{2\pi}{2n+1}\right)_\infty \lambda_n(\bar{x}) + \mathcal{O}\left(\omega_r(f; n^{-1})_\infty\right),$$

where $t_n f \in T_n$ interpolates the 2π-periodic continuous function f in the equidistant nodes $\bar{x} = (x_{-n}, \ldots, x_n)$, $x_k = 2k\pi/(2n+1)$, $k = -n, \ldots, n$, and $\lambda_n(\bar{x})$ is

the Lebesgue constant for the trigonometric Lagrange interpolation. For similar estimates in uniform norm, concerning the approximation by the partial sums of the Fourier series, one can refer to [2] and [4].

The trigonometric analogue of the Taylor's formula will allow us to derive a point-wise estimate of the error $f(x) - L_n(f, x)$ for a smooth f. We need to introduce several notations to state that result. We define the differential operators

$$D_j = \left(\frac{d}{dx}\right)^2 + j^2 I, \quad j = 1, 2, \ldots, \tag{1.3}$$

where I is the identity. We also put

$$\tilde{D}_{n+1} = D_n \cdots D_1 \frac{d}{dx},$$

$$\hat{D}_{n0} = D_1 \cdots D_n,$$

$$\hat{D}_{nk} = D_1 \cdots D_{k-1} D_{k+1} \cdots D_n, \quad k = 1, \ldots, n.$$

Let us observe that $\tilde{D}_{n+1}g = 0$, $g \in C^{2n+1}[a, b]$, if and only if $g \in T_n$ in $[a, b]$. The following trigonometric analogue of the Taylor's formula holds true (see [5, §10.8]).

Theorem 1 (Taylor's trigonometric formula). *Let $f \in C^{2n+1}(\Delta_c)$, where Δ_c is any of the intervals $[c, c + \delta]$, $[c - \delta, c]$ or $[c - \delta, c + \delta]$ for $c \in \mathbf{R}$ and $\delta > 0$, and let also*

$$\tau_{n,c}(f, x) = \frac{\hat{D}_{n0}f(c)}{(n!)^2} + 2 \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(n-k)!\,(n+k)!}$$

$$\times \left[(k^2 \hat{D}_{nk}f(c) - \hat{D}_{n0}f(c)) \cos k(x - c) + k\hat{D}_{nk}f'(c) \sin k(x - c)\right]. \tag{1.4}$$

Then $\tau_{n,c}f \in T_n$, $\tau_{n,c}^{(s)}(f, c) = f^{(s)}(c)$, $s = 0, 1, \ldots, 2n$, and for $x \in \Delta_c$ we have

$$f(x) = \tau_{n,c}(f, x) + \frac{1}{n!\,(2n-1)!!} \int_c^x (1 - \cos(x - t))^n \tilde{D}_{n+1}f(t)\, dt. \tag{1.5}$$

Let $-\pi \le x_0 < \cdots < x_{2n} < \pi$ be arbitrary nodes. Let us denote by $t_n(f, x)$ the unique trigonometric polynomial of degree n, which interpolates f in those nodes. Then the theorem above easily implies a point-wise estimate of the error $f(x) - t_n(f, x)$ for a smooth function f.

Proposition 2. *Let $f \in C^{2n+1}[-\pi, \pi]$. Then*

$$f(x) - t_n(f, x) = \frac{1}{n!\,(2n-1)!!} \int_{-\pi}^{\pi} K(x, t)\tilde{D}_{n+1}f(t)\, dt, \quad x \in [-\pi, \pi],$$

where

$$K(x, t) = \left(1 - \cos[(x - t)_+]\right)^n - \sum_{k=0}^{2n} \left(1 - \cos[(x_k - t)_+]\right)^n t_{nk}(x)$$

and $(x - t)_+ = \max\{x - t, 0\}$.

The contents of the paper are organized as follows. In Section 2 we collect few auxiliary results, which are necessary for the proof of the Taylor's trigonometric formula, presented in Section 3. Finally, in the last section we derive point-wise estimates of the error in the trigonometric interpolation and in the approximation by convolutional linear operators.

2. AUXILIARY RESULTS

Let $[a, b]$ be a finite interval such that $0 \in [a, b]$. We define the convolutional operator, known as Duhamel's convolution, $\circledast : L_1[a, b] \times L_1[a, b] \to L_1[a, b]$,

$$f \circledast g(x) := \int_0^x f(x - t) g(t) \, dt.$$

It is easy to verify that it possesses the properties:

1. $f \circledast g = g \circledast f$;

2. $f \circledast (g + h) = f \circledast g + f \circledast h$;

3. $f \circledast (g \circledast h) = (f \circledast g) \circledast h$.

Next we introduce a number of notations. We put $\varphi_n(x) = \sin nx$, $n = 1, 2, \ldots$, and $\Phi_n = \varphi_1 \circledast \cdots \circledast \varphi_n$, $\widetilde{\Phi}_n = \Phi_n \circledast 1$, $\widehat{\Phi}_n = \widetilde{\Phi}_n \circledast 1$. The propositions bellow contain some of the properties of Φ_n, $\widetilde{\Phi}_n$ and $\widehat{\Phi}_n$, but first we prove the following simple lemma.

Lemma 3. *Any function of the form*

$$f(x) = cx + a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \tag{2.1}$$

has at most $2n + 1$ zeroes in $[-\pi, \pi)$, counting the multiplicities, that is, $x, 1, \cos x$, $\sin x, \ldots, \cos nx, \sin nx$ is an extended Chebyshev system in $[-\pi, \pi)$. Hence, for any choice of $-\pi \le x_1 < \cdots < x_m < \pi$ and positive integers $\nu_1, \ldots, \nu_m$ with $\nu_1 + \cdots + \nu_m = 2n + 1$ there exists only one function of the form (2.1) with a fixed c for which x_k is a zero of multiplicity ν_k, $k = 1, \ldots, m$.

Proof. It is enough to prove the first part of the statement. We follow a standard argument assuming the opposite and making use of the well-known Rolle's theorem. So, let us assume that $f(x)$ has at least $2n + 2$ zeroes in $[-\pi, \pi)$, counting the multiplicities. Then $f'(x)$ has at least $2n + 1$ zeroes in $[-\pi, \pi)$, counting the multiplicities. But $f'(x)$ is a trigonometric polynomial of degree n and therefore it has at most $2n$ zeroes in $[-\pi, \pi)$, counting the multiplicities. This contradiction verifies the statement of the lemma. $\square$

Proposition 4. *We have*

(i) $D_n \Phi_n = n\Phi_{n-1}$ *and* $D_n \widehat{\Phi}_n = n\widehat{\Phi}_{n-1}$ *for* $n = 2, 3, \ldots$;

(ii) $\Phi_n(x) = c_n \sin x (1 - \cos x)^{n-1}$, *where* $c_n = \dfrac{n}{(2n-1)!!}$, $n = 1, 2 \ldots$;

(iii) $\widetilde{\Phi}_n(x) = \dfrac{1}{(2n-1)!!}(1 - \cos x)^n$;

(iv) $\widehat{\Phi}_n(x) = \dfrac{x}{n!} + \displaystyle\sum_{k=1}^{n} b_{nk} \sin kx$,

where $\{b_{nk}\}$ *is the unique solution of the linear system*

$$\left| \begin{array}{l} \displaystyle\sum_{k=1}^{n} k b_{nk} = -\frac{1}{n!}, \\[3mm] \displaystyle\sum_{k=1}^{n} k^s b_{nk} = 0, \ s = 3, 5, \ldots, 2n-1. \end{array} \right.$$

Proof. The first statement of the proposition follows by differentiation of the recursion relation $\Phi_n = \varphi_n \circledast \Phi_{n-1}$. Namely, we have

$$\left(\frac{d}{dx}\right)^2 \Phi_n(x) = \left(\frac{d}{dx}\right)^2 \int_0^x \sin n(x-t)\Phi_{n-1}(t)\, dt = n\frac{d}{dx}\int_0^x \cos n(x-t)\Phi_{n-1}(t)\, dt$$

$$= n\Phi_{n-1}(x) - n^2 \int_0^x \sin n(x-t)\Phi_{n-1}(t)\, dt$$

$$= n\Phi_{n-1}(x) - n^2 \Phi_n(x).$$

Thus we have got $D_n \Phi_n = n\Phi_{n-1}$. If we put $e_1(x) = x$, then $\widehat{\Phi}_n = \widetilde{\Phi}_n \circledast 1 = \Phi_n \circledast 1 \circledast 1 = \Phi_n \circledast e_1$. Therefore $\widehat{\Phi}_n$ satisfies the same recursion relation as Φ_n with $\widehat{\Phi}_1(x) = x - \sin x$ instead of $\Phi_1(x) = \sin x$. Hence we get $D_n \widehat{\Phi}_n = n\widehat{\Phi}_{n-1}$. This completes the proof of (i).

To verify (ii), we consider the sequence of trigonometric polynomials

$$P_n(x) = c_n \sin x (1 - \cos x)^{n-1}, \quad c_n = \frac{n}{(2n-1)!!}, \quad n \geq 1.$$

We shall show that it satisfies the same recursion relation as Φ_n in (i) and $P_n(0) = 0 = \Phi_n(0)$, $P_n'(0) = 0 = \Phi_n'(0)$, $n \geq 2$. Hence, as $P_1 = \Phi_1$, we have $P_n = \Phi_n$, $n \geq 2$. For $n \geq 2$

$$P_n''(x) = c_n \left(\sin x (1 - \cos x)^{n-1}\right)''$$

$$= c_n \sin x (1 - \cos x)^{n-2}(n^2 - 3n + 1 + n^2 \cos x).$$

Consequently,

$$\begin{aligned}
D_n P_n(x) &= P_n''(x) + n^2 P_n(x) \\
&= c_n \sin x (1 - \cos x)^{n-2}(n^2 - 3n + 1 + n^2 \cos x) \\
&\qquad\qquad\qquad\qquad + n^2 c_n \sin x (1 - \cos x)^{n-1} \\
&= c_n(2n^2 - 3n + 1)\sin x (1 - \cos x)^{n-2} \\
&= n c_{n-1} \sin x (1 - \cos x)^{n-2} \\
&= n P_{n-1}(x).
\end{aligned}$$

We get (iii) by integrating (ii).

It remains to verify (iv). From (i) it follows $\tilde{D}_{n+1}\widehat{\Phi}_n = n!$. Consequently, $\widehat{\Phi}_n(x) = x/n! + a_0 + \sum_{k=1}^n (a_{nk}\cos kx + b_{nk}\sin kx)$ for some constants a_{nk}, b_{nk}. Assertion (iii) implies that $\tilde{\Phi}_n$ is an even function, therefore $\widehat{\Phi}_n$ is an odd one. This implies that $\widehat{\Phi}_n(x) = x/n! + \sum_{k=1}^n b_{nk}\sin kx$ for some $b_{nk} \in \mathbf{R}$. Next we have $\widehat{\Phi}_n'(0) = \tilde{\Phi}_n(0) = 0$, which implies

$$\sum_{k=1}^n k b_{nk} = -\frac{1}{n!}.$$

It is easy to see that $\widehat{\Phi}_n^{(s)}(0) = 0$, $s = 2, \ldots, 2n - 1$, as well. For s even this is obvious. For s odd we can verify it, for instance, by induction on n. For $n = 1$ the statement is trivial as we have shown above. We assume that $\widehat{\Phi}_n^{(s)}(0) = 0$, $s = 1, \ldots, 2n - 1$, and shall verify it for $n + 1$ in the place of n. We differentiate in x the equality $D_{n+1}\widehat{\Phi}_{n+1}(x) = (n+1)\widehat{\Phi}_n(x)$ and get for $s = 1, \ldots, 2n - 1$

$$\widehat{\Phi}_{n+1}^{(s+2)}(x) + (n+1)^2\widehat{\Phi}_{n+1}^{(s)}(x) = (n+1)\widehat{\Phi}_n^{(s)}(x).$$

Then, putting $x = 0$, we get $\widehat{\Phi}_{n+1}^{(s)}(0) = 0$ consecutively for $s = 3, 5, \ldots, 2n + 1$, which is what we had to show. Now

$$\sum_{k=1}^n k^s b_{nk} = 0, \quad s = 3, 5, \ldots, 2n - 1,$$

follows from $\widehat{\Phi}_n^{(s)}(0) = 0$, $s = 3, \ldots, 2n - 1$ ($n > 1$). In passing, let us note that the linear system

$$\left|\begin{aligned}
&\sum_{k=1}^n k b_{nk} = -\frac{1}{n!}, \\
&\sum_{k=1}^n k^s b_{nk} = 0, \quad s = 3, 5, \ldots, 2n - 1
\end{aligned}\right.$$

has a unique solution due to Lemma 3. This completes the proof of (iv). $\square$

The following representation of $\widehat{\Phi}_n(x)$ has been pointed out to the author by K. G. Ivanov.

Proposition 5. *The following formula holds :*

$$\widehat{\Phi}_n(x) = \frac{1}{n!}\left(x - \sum_{k=1}^{n}\frac{(k-1)!}{(2k-1)!!}\sin x(1-\cos x)^{k-1}\right). \qquad (2.2)$$

Proof. We just write for $n \geq 1$

$$J_n(x) := \int_0^x \sin^{2n} t\, dt = -\int_0^x \sin^{2n-1} t\, d\cos t$$

$$= -\sin^{2n-1} x \cos x + (2n-1)\int_0^x \cos^2 t \sin^{2(n-1)} t\, dt$$

$$= -\sin^{2n-1} x \cos x + (2n-1)\int_0^x \sin^{2(n-1)} t\, dt - (2n-1)\int_0^x \sin^{2n} t\, dt.$$

Therefore

$$J_n(x) = -\sin^{2n-1} x \cos x + (2n-1)J_{n-1}(x) - (2n-1)J_n(x).$$

Hence we get the recursion relation

$$J_n(x) = -\frac{1}{2n}\sin^{2n-1} x \cos x + \frac{2n-1}{2n}J_{n-1}(x), \quad n \geq 1.$$

Consequently, noting that $J_0(x) = x$, we get

$$J_n(x) = \frac{1}{2n}\left(\frac{(2n-1)!!}{(2n-2)!!}x - \sin^{2n-1} x \cos x\right.$$

$$\left. - \sum_{l=1}^{n-1}\frac{(2n-1)(2n-3)\cdots(2n-2l+1)}{(2n-2)(2n-4)\cdots(2n-2l)}\sin^{2n-2l-1} x \cos x\right)$$

$$= \frac{1}{2n}\left(\frac{(2n-1)!!}{(2n-2)!!}x - \frac{1}{2}\sin^{2(n-1)} x \sin 2x\right.$$

$$\left. - \frac{(2n-1)!!}{2(2n-2)!!}\sum_{l=1}^{n-1}\frac{(2n-2l-2)!!}{(2n-2l-1)!!}\sin^{2(n-l-1)} x \sin 2x\right)$$

$$= \frac{(2n-1)!!}{(2n)!!}\left(x - \frac{(2n-2)!!}{2(2n-1)!!}\sin^{2(n-1)} x \sin 2x\right.$$

$$\left. - \frac{1}{2}\sum_{l=1}^{n-1}\frac{(2n-2l-2)!!}{(2n-2l-1)!!}\sin^{2(n-l-1)} x \sin 2x\right)$$

$$= \frac{(2n-1)!!}{(2n)!!}\left(x - \frac{1}{2}\sum_{l=0}^{n-1}\frac{(2n-2l-2)!!}{(2n-2l-1)!!}\sin^{2(n-l-1)} x \sin 2x\right)$$

$$= \frac{(2n-1)!!}{(2n)!!}\left(x - \frac{1}{2}\sum_{k=1}^{n}\frac{(2k-2)!!}{(2k-1)!!}\sin^{2(k-1)} x \sin 2x\right)$$

$$= \frac{(2n-1)!!}{(2n)!!}\left(x - \frac{1}{2}\sum_{k=1}^{n}\frac{(k-1)!}{(2k-1)!!}2^{k-1}\sin^{2(k-1)}x\sin 2x\right)$$

$$= \frac{(2n-1)!!}{2^{n+1}n!}\left(2x - \sum_{k=1}^{n}\frac{(k-1)!}{(2k-1)!!}(1-\cos 2x)^{k-1}\sin 2x\right).$$

Thus we have shown

$$J_n(x) = \frac{(2n-1)!!}{2^{n+1}n!}\left(2x - \sum_{k=1}^{n}\frac{(k-1)!}{(2k-1)!!}\sin 2x(1-\cos 2x)^{k-1}\right). \qquad (2.3)$$

To finish the proof, we just write

$$\widehat{\Phi}_n(x) = \frac{1}{(2n-1)!!}\int_0^x (1-\cos t)^n\, dt = \frac{2^{n+1}}{(2n-1)!!}\int_0^{x/2}\sin^{2n} t\, dt$$

$$= \frac{2^{n+1}}{(2n-1)!!}J_n(x/2).$$

Hence, making use of (2.3), we get (2.2). □

Let $[a, b]$ be a finite interval such that $0 \in [a, b]$. In [1] we have proved that $\mathcal{F}_n : C[a, b] \to C[a, b]$ can be represented in the form

$$\mathcal{F}_n = A_1 \cdots A_n,$$

where the bounded linear operators $A_j : C[a, b] \to C[a, b]$, $j = 1, 2, \ldots$, are defined by

$$A_j(f, x) := f(x) + j^2\int_0^x (x-t)f(t)\, dt, \quad j = 1, 2, \ldots$$

In the above mentioned investigation we have also shown the following assertion.

Proposition 6. *The bounded linear operator A_j is invertible and*

$$A_j^{-1}(g, x) = g(x) - j\int_0^x \sin j(x-t)g(t)\, dt.$$

Hence

$$A_j^{-1}(g, x) = \frac{1}{j}\int_0^x \sin j(x-t)g''(t)\, dt$$

for $g \in C^2[a, b]$ with $g(0) = g'(0) = 0$.

3. THE PROOF OF THE TAYLOR'S TRIGONOMETRIC FORMULA

Now we are ready to prove formula (1.5).

Proof of Theorem 1 It is enough to prove the assertion of the theorem for $c = 0$. Hence it will follow for any $c \in \mathbf{R}$ by translation. Let $\tau(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$ be the unique trigonometric polynomial of degree at most n, which interpolates f in $x = 0$ with multiplicity $2n + 1$, i.e., $\tau^{(s)}(0) = f^{(s)}(0)$ for $s = 0.1, \ldots, 2n$. Using

$$D_j \cos kx = (j^2 - k^2) \cos kx \quad \text{and} \quad D_j \sin kx = (j^2 - k^2) \sin kx,$$

we get

$$(n!)^2 a_0 = \widehat{D}_{n0}\tau(0) = \widehat{D}_{n0}f(0),$$

$$(-1)^{k-1} \frac{(n-k)!\,(n+k)!}{2k^2} a_k + \frac{(n!)^2}{k^2} a_0 = \widehat{D}_{nk}\tau(0) = \widehat{D}_{nk}f(0), \quad k = 1, \ldots, n,$$

$$(-1)^{k-1} \frac{(n-k)!\,(n+k)!}{2k} b_k = \widehat{D}_{nk}\tau'(0) = \widehat{D}_{nk}f'(0), \quad k = 1, \ldots, n.$$

Hence $\tau_{n,0}(f, x) = \tau(x)$.

It remains to consider the remainder $r_n(x) = f(x) - \tau_{n,0}(f, x)$. Let us put for the sake of brevity

$$F(x) = \int_0^x \left(\int_0^{t_1} \cdots \left(\int_0^{t_{2n}} \widetilde{D}_{n+1}f(t_{2n+1}) \, dt_{2n+1} \right) \cdots dt_2 \right) dt_1.$$

Obviously, $F \in C^{2n+1}(\Delta_0)$ and $F^{(s)}(0) = 0$, $s = 0, 1, \ldots, 2n$. Now $r_n(x) = f(x) - \tau_{n,0}(f, x)$ implies $\widetilde{D}_{n+1}r_n(x) = \widetilde{D}_{n+1}f(x)$, $x \in \Delta_0$. We have proved in [1] that

$$(\mathcal{F}_n g)^{(2n+1)} = \widetilde{D}_{n+1}g, \quad g \in C^{2n+1}(\Delta_0).$$

Therefore $(d/dx)^{2n+1} \mathcal{F}_n(r_n, x) = \widetilde{D}_{n+1}r_n(x)$, $x \in \Delta_0$. Hence, making use of $r_n^{(s)}(0) = 0$, $s = 0, 1, \ldots, 2n$, we get $\mathcal{F}_n(r_n, x) = F(x)$, $x \in \Delta_0$, that is,

$$A_1 \cdots A_n r_n = F. \tag{3.1}$$

Proposition 6 states for $g \in C^2(\Delta_0)$ with $g(0) = g'(0) = 0$ that

$$A_j^{-1} g = \frac{1}{j} \varphi_j \circledast g''. \tag{3.2}$$

Simple calculations yield for $g \in C^2(\Delta_0)$ with $g(0) = g'(0) = 0$

$$(\Phi_k \circledast g)'' = \Phi_k \circledast g'', \tag{3.3}$$

and for any $g \in C(\Delta_0)$

$$\Phi_k \circledast g(0) = (\Phi_k \circledast g)'(0) = 0. \tag{3.4}$$

Now (3.1) and (3.2) for $j = 1$ imply

$$A_2 \cdots A_n r_n = \varphi_1 \circledast F'' = \Phi_1 \circledast F''.$$

Next, applying again (3.2) (for $j = 2$), using (3.4) (for $k = 1$), and then (3.3) (for $k = 1$), we have

$$A_3 \cdots A_n r_n = \frac{1}{2} \varphi_1 \circledast \varphi_2 \circledast F^{(4)} = \frac{1}{2} \Phi_2 \circledast F^{(4)}.$$

Proceeding in this way, we finally get

$$r_n = \frac{1}{n!} \Phi_n \circledast F^{(2n)}. \tag{3.5}$$

To finish the proof, we write

$$
\begin{aligned}
r_n(x) &= \frac{1}{n!} \int_0^x \left(\Phi_n(x - t) \int_0^t \widetilde{D}_{n+1} f(s) \, ds \right) dt \\
&= -\frac{1}{n!} \int_0^x \left(\int_0^t \widetilde{D}_{n+1} f(s) \, ds \right) d\widetilde{\Phi}_n(x - t) \\
&= \frac{1}{n!} \int_0^x \widetilde{\Phi}_n(x - t) \widetilde{D}_{n+1} f(t) \, dt \\
&= \frac{1}{n!} \widetilde{\Phi}_n \circledast \widetilde{D}_{n+1} f(x).
\end{aligned}
$$

This completes the proof of the theorem as Proposition 4 (iii) states $\widetilde{\Phi}_n(x) = 1/(2n-1)!! \, (1 - \cos x)^n$. $\square$

Remark 7. An estimate of the remainder. (Again we discuss the case $c = 0$.) The mean value theorem implies

$$r_n(x) = \frac{\widetilde{D}_{n+1} f(\xi_x)}{n! \, (2n-1)!!} \int_0^x (1 - \cos t)^n \, dt, \quad x \in \Delta_0, \tag{3.6}$$

where $\xi_x \in \Delta_0$ depends on x. Hence

$$|r_n(x)| \leq \frac{\|\widetilde{D}_{n+1} f\|_{\infty(\Delta_0)}}{n! \, (2n-1)!!} \left| \int_0^x (1 - \cos t)^n \, dt \right|, \quad x \in \Delta_0. \tag{3.7}$$

Now, using the simple inequality $1 - \cos x \leq x^2/2$, we get

$$|r_n(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!} \|\widetilde{D}_{n+1} f\|_{\infty(\Delta_0)}, \quad x \in \Delta_0. \tag{3.8}$$

Formula (1.5) can be useful in expressing the error in approximation by linear operators that preserves trigonometric polynomials up to a given degree. Indeed, let $L_n : C[-\pi, \pi] \to C[-\pi, \pi]$ be such that $L_n f = f$ if $f \in T_n$ and let $f \in C^{2n+1}[-\pi, \pi]$. Then we have

$$f - L_n f = (I - L_n) r_n f, \tag{4.1}$$

where

$$r_n(f, x) = \frac{1}{n! \, (2n-1)!!} \int_c^x (1 - \cos(x - t))^n \widetilde{D}_{n+1} f(t) \, dt$$

for some fixed $c \in [-\pi, \pi]$.

Let $-\pi \le x_0 < \cdots < x_{2n} < \pi$ be arbitrary nodes. Then, as it is known, there exists a unique trigonometric polynomial $t_n(f, x)$ of degree n such that $t_n(f, x_k) = f(x_k)$, $k = 0, \ldots, 2n$. It can be represented in the form

$$t_n(f, x) = \sum_{k=0}^{2n} f(x_k) t_{nk}(x), \tag{4.2}$$

where

$$t_{nk}(x) = \frac{\displaystyle\prod_{j=0, j \ne k}^{2n} \sin \frac{x - x_j}{2}}{\displaystyle\prod_{j=0, j \ne k}^{2n} \sin \frac{x_k - x_j}{2}}. \tag{4.3}$$

Now the considerations in the beginning of this section and (1.5) with $c = -\pi$ easily yield Proposition 2. That proposition implies the following estimates of the error $f(x) - t_n(f, x)$ for smooth functions f.

Corollary 8. *Let $f \in C^{2n+1}[-\pi, \pi]$. Then we have for $x \in [-\pi, \pi]$*

(i) $|f(x) - t_n(f, x)| \le \dfrac{\pi \mu(\bar{x}) \|\widetilde{D}_{n+1} f\|_\infty}{2^n (n-1)! \, (2n-1)!!} |(x - x_0) \ldots (x - x_{2n})|,$

where

$$\mu(\bar{x}) = \sum_{k=0}^{2n} \left(\prod_{j=0, j \ne k}^{2n} \left| \sin \frac{x_k - x_j}{2} \right| \right)^{-1}.$$

(ii) $|f(x) - t_n(f, x)| \le \dfrac{2^{n+1} \pi^2 \mu(\bar{x}) \|\widetilde{D}_{n+1} f\|_\infty}{a \, (n-1)! \, (2n-1)!!} \left| \sin \dfrac{x - x_0}{2} \cdots \sin \dfrac{x - x_{2n}}{2} \right|$

for nodes $-\pi + a \le x_0 < \cdots < x_{2n} \le \pi - a$, $a \in (0, \pi)$.

Proof. The assertions of the corollary follow easily from the estimate

$$\left| (1 - \cos[(x - t)_+])^n - (1 - \cos[(x_k - t)_+])^n \right|$$
$$\le n 2^{n-1} |\cos[(x_k - t)_+] - \cos[(x - t)_+]| \tag{4.4}$$

and the relation

$$\cos[(x_k-t)_+]-\cos[(x-t)_+] = \begin{cases} -2\sin\dfrac{x+x_k-2t}{2}\sin\dfrac{x_k-x}{2}, & t\le x, x_k, \\[2mm] 2\sin^2\dfrac{x-t}{2}, & x_k\le t\le x, \\[2mm] -2\sin^2\dfrac{x_k-t}{2}, & x\le t\le x_k, \\[2mm] 0, & t\ge x. x_k. \end{cases} \quad (4.5)$$

Now (4.4), (4.5) and the inequality $|\sin x|\le |x|$ imply

$$|(1-\cos[(x-t)_+])^n - (\cos[(x_k-t)_+])^n| \le n2^{n-1}|x-x_k|,$$

therefore, using again the inequality $|\sin x|\le |x|$ and the fact that $\sum_{k=0}^{2n} t_{nk}(x)\equiv 1$, we get for any x and t

$$|K(x,t)|\le n2^{n-1}\sum_{k=0}^{2n}|x-x_k||t_{nk}(x)| \le n2^{-n-1}\mu(\bar{x})|x-x_0|\cdots|x-x_{2n}|.$$

Hence assertion (i) follows. To verify (ii), we just have to notice that if $-\pi+a\le x_0<\cdots<x_{2n}\le \pi - a$, where $a\in(0,\pi)$, and $x\in[-\pi,\pi]$, then

$$\sin\frac{x-t}{2} \le \frac{\sin\dfrac{x-x_k}{2}}{\sin\dfrac{a}{2}}, \quad x_k\le t\le x, \quad \text{and} \quad \sin\frac{x_k-t}{2} \le \frac{\sin\dfrac{x_k-x}{2}}{\sin\dfrac{a}{2}}, \quad x\le t\le x_k.$$

These two estimates, (4.4), (4.5) and the inequality $|\sin x|\ge (2/\pi)|x|$, $|x|\le \pi/2$, yield for $x\in[-\pi,\pi]$ and any t

$$|(1-\cos[(x-t)_+])^n - (\cos[(x_k-t)_+])^n| \le \frac{n2^n\pi}{a}\left|\sin\frac{x-x_k}{2}\right|,$$

which, on its turn, implies for $x\in[-\pi,\pi]$ and any t

$$|K(x,t)| \le \frac{n2^n\pi}{a}\sum_{k=0}^{2n}\left|\sin\frac{x-x_k}{2}\right||t_{nk}(x)|$$
$$= \frac{n2^n\pi\mu(\bar{x})}{a}\left|\sin\frac{x-x_0}{2}\right|\cdots\left|\sin\frac{x-x_{2n}}{2}\right|.$$

Hence assertion (ii) follows. □

Remark 9. Our conjecture is that for any fixed $x'\in[-\pi,\pi]$ the kernel $K(x',t)$ does not change its sign in $[-\pi,\pi]$. If that is true, then the mean value theorem implies the Lagrange-type estimate

$$f(x)-t_n(f,x) = \frac{\tilde{D}_{n+1}f(\xi_x)}{(n!)^2}\omega(x), \quad x\in[-\pi,\pi],$$

where $f \in C^{2n+1}[-\pi, \pi]$, and

$$w(x) = x + a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

is the only function of this form, which vanishes in the nodes $\{x_k\}_{k=0}^{2n}$ and has no other zeroes in $[-\pi, \pi)$. Actually,

$$w(x) = x - \sum_{k=0}^{2n} x_k t_{nk}(x).$$

Let the bounded linear operator $L_n : C^*[-\pi, \pi] \to C^*[-\pi, \pi]$ be of the form

$$L_n(f, x) = \mathcal{M}_n * f(x) := \int_{-\pi}^{\pi} \mathcal{M}_n(x - t)f(t)\,dt, \qquad (4.6)$$

where $\mathcal{M}_n \in L_1^*[-\pi, \pi]$. For any fixed $t \in [-\pi, \pi]$ we define the 2π-periodic function $\rho_t : \mathbf{R} \to \mathbf{R}$ by

$$\rho_t(x) := 1 - \cos[(x - 2k\pi - t)_+], \quad x \in [(2k-1)\pi, (2k+1)\pi), \ k \in \mathbf{Z}.$$

It is quite easy to verify the following assertion.

Proposition 10. *Let $f \in C^{2n+1}[-\pi, \pi]$ be 2π-periodic. Let also the bounded linear operator L_n, defined by (4.6), preserve the trigonometric polynomials of degree n. Then*

$$f(x) - L_n(f, x) = \frac{1}{n!\,(2n-1)!!} \int_{-\pi}^{\pi} [\rho_t^n(x) - \mathcal{M}_n * \rho_t^n(x)]\widetilde{D}_{n+1}f(t)\,dt.$$

Proof. Making use of formula (1.5) with $c = -\pi$ and changing the order of integration after that, we get easily the estimate

$$f(x) - L_n(f, x) = \frac{1}{n!\,(2n-1)!!} \int_{-\pi}^{\pi} (1 - \cos[(x-t)_+])^n \widetilde{D}_{n+1}f(t)\,dt$$

$$- \frac{1}{n!\,(2n-1)!!} \int_{-\pi}^{\pi} \mathcal{M}_n(x-t)\left(\int_{-\pi}^{\pi}(1-\cos[(t-u)_+])^n \widetilde{D}_{n+1}f(u)\,du\right)dt$$

$$= \frac{1}{n!\,(2n-1)!!} \int_{-\pi}^{\pi} \left((1 - \cos[(x-t)_+])^n\right.$$

$$\left. - \int_{-\pi}^{\pi} \mathcal{M}_n(x-u)(1-\cos[(u-t)_+])^n\,du\right)\widetilde{D}_{n+1}f(t)\,dt.$$

Thus the proof is completed. $\square$

Immediately, Proposition 10 yields

Corollary 11. *Let $f \in C^{2n+1}[-\pi, \pi]$ be 2π-periodic. Let also the bounded linear operator L_n, defined by (4.6), preserve the trigonometric polynomials of degree n. Then*

$$\|f - L_n f\|_\infty \leq \frac{2^{n+1}\pi}{n!\,(2n-1)!!}(1 + \|\mathcal{M}_n\|_1)\|\widetilde{D}_{n+1}f\|_\infty.$$

REFERENCES

1. Draganov, B. R. A new modulus of smoothness for trigonometric polynomial approximation. *East J. Approx.*, **8**, 2002, 465–499.

2. Natanson, G. I. Some cases when Fourier series yields the best order of approximation. *Dokl. Akad. Nauk USSR*, **183**, 1968, 1254-1257 (in Russian).

3. Nevai, G. P. On the deviation of trigonometric interpolation sums. *Acta Math. Acad. Sci. Hungar.*, **23**, 1972, 203-205 (in Russian).

4. Nevai, G. P. Remarks to a theorem of G. I. Natanson. *Acta Math. Acad. Sci. Hungar.*, **23**, 1972, 219-221 (in Russian).

5. Schumaker, L. L. Spline functions: basic theory. Wiley-Interscience, New York. 1981.

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