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MAXIMAL DEPTHS OF BOOLEAN FUNCTIONS

DIMITER SKORDEV

Given any Boolean function, there is an upper bound of its depths with respect to arbitrary complete sets of such functions. We prove the algorithmic computability of the largest of these depths.

Keywords: Boolean function, complete set, Post theorem, maximal depth, algorithmic computability

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1. INTRODUCTION

The depths of the Boolean functions are sometimes used for measuring their complexity (especially in the case when parallel computations are considered). Actually, there are at least two variants of the notion of depth. The difference comes from the presence or absence of the possibility to use 0's and 1's "for free" in the computations. The first of these options is chosen for example in [1]. The notion of depth is defined there in Section 1.3 through a corresponding notion of Boolean circuit, and Section 1.4 shows that an approach through Boolean formulas would yield the same values of the depths. The other variant of the notion, also current in the literature, can be defined in a quite similar way, but without the possibility to use the constants 0 and 1 as predecessors of the gates of the considered circuits.

The gates of each Boolean circuit have as their types Boolean functions belonging to some given set, which usually is chosen to be complete.[1] Therefore the

[1] In fact, the notion of completeness also splits into two ones - the weaker notion corresponds to possible using of 0's and 1's "for free", whereas the stronger one corresponds to the case when there is no such possibility.

depth of a function f depends not only on f, but also on the choice of this set of functions. When we compare the depths of two Boolean functions, the result may also depend on the choice in question. For example, the implication has a smaller depth than the equivalence with respect to the set consisting of negation and conjunction, but the situation is the opposite with respect to the set, whose elements are the constant 1, addition modulo 2 and conjunction (of course, if the constants can be used "for free", the constant 1 may be omitted from the second of these sets). To get a complexity measure depending only on the function f, we shall look for the depth of f in the worst case, i.e. in the case when the depth is maximally large.

2. SOME DEFINITIONS

To avoid reasoning about Boolean circuits or Boolean formulas, we shall define the notion of depth (and also of completeness) in another equivalent way. Suppose Ω is a set of Boolean functions[2]. We define infinite sequences $\Omega^{(0)}, \Omega^{(1)}, \Omega^{(2)}, \dots$ and $\Omega^{[0]}, \Omega^{[1]}, \Omega^{[2]}, \dots$ of sets of Boolean functions as follows:

- $\Omega^{(0)}$ is the set of all Boolean functions of the form

  $$g(x_1, \dots, x_n) = x_k, \ \ n = 1, 2, 3, \dots, \ k = 1, 2, \dots, n$$

  (*the projection functions*), whereas $\Omega^{[0]}$ consists of these functions and also all constant Boolean functions;

- $\Omega^{(r+1)}$ is obtained by adding to $\Omega^{(r)}$ all functions of the form

  $$g(x_1, \dots, x_n) = f(g_1(x_1, \dots, x_n), \dots, g_m(x_1, \dots, x_n))$$

  with f belonging to Ω and $g_1, \dots, g_m$ belonging to $\Omega^{(r)}$, and $\Omega^{[r+1]}$ is obtained similarly, but with a replacement of $\Omega^{(r)}$ by $\Omega^{[r]}$.

We shall call the set Ω *strongly complete* if each Boolean function belongs to $\Omega^{(r)}$ for some non-negative integer r. The set Ω will be called *weakly complete* if each Boolean function belongs to $\Omega^{[r]}$ for some non-negative integer r. Since $\Omega^{(r)} \subseteq \Omega^{[r]}$ for all Ω and r, any strongly complete set is also weakly complete.

Let h be a Boolean function. If Ω is a strongly complete set of Boolean functions, then the smallest r such that $h \in \Omega^{(r)}$ will be called *the strong depth of h with respect to* Ω, and it will be denoted by ${}^{s}D_{\Omega}(h)$. Similarly, if Ω is a weakly complete set of Boolean functions, then the smallest r such that $h \in \Omega^{[r]}$ will be called *the weak depth of h with respect to* Ω, and it will be denoted by ${}^{w}D_{\Omega}(h)$. We note that ${}^{s}D_{\Omega}(h) \geq {}^{w}D_{\Omega}(h)$ for any strongly complete set Ω (due to the inclusion $\Omega^{(r)} \subseteq \Omega^{[r]}$).

[2]We shall consider Boolean functions only of a non-zero number of arguments. In particular, the constants will be regarded as such ones too.

There is an easy reduction of the notions of weak completeness and weak depth to the notions of strong completeness and strong depth, respectively.

Lemma 2.1. *Let Ω be a set of Boolean functions, and let Ω' consist of the one-argument constants* 0 *and* 1 *and of all Boolean functions (including the ones from Ω) that have the form*

$$\lambda x_1 \lambda x_2 \dots \lambda x_n . f(c_{01}, \dots, c_{0k_0}, x_1, c_{11}, \dots, c_{1k_1}, x_2, \dots, x_n, c_{n1}, \dots, c_{nk_n})$$

with f in Ω and c_{ij} in $\{0, 1\}$. Then $\Omega^{[r]} = \Omega'^{(r)}$ for $r = 1, 2, 3, \dots$

Proof. By induction on r □.

Corollary 2.1. *Let Ω be a set of Boolean functions, and let Ω' be defined as in Lemma* 2.1. *Then Ω is weakly complete iff Ω' is strongly complete, and in such a case the equality*

$$^{w}D_{\Omega}(h) = {}^{s}D_{\Omega'}(h) \tag{2.1}$$

holds for any non-constant Boolean function h.

Of course, the equality (2.1) does not hold for constant functions, since they have weak depth 0 with respect to any weakly complete set Ω, whereas their strong depths with respect to the corresponding set Ω' will be equal to 1.

By a well-known theorem of Emil Post, the strongly complete sets of Boolean functions can be characterized as follows: a set Ω of Boolean functions is strongly complete iff there are in Ω at least one function not preserving 0, at least one function not preserving 1, at least one function that is not self-dual, at least one function that is not monotonically increasing and at least one non-linear function. Hence, by Corollary 2.1, a set Ω of Boolean functions is weakly complete iff there are in Ω at least one function that is not monotonically increasing and at least one non-linear function.

Remark 2.1. Whenever a finite strongly complete set Ω of Boolean functions and a positive integer n are given, one can consecutively find lists of all n-argument functions in the sets $\Omega^{(r)}$ for $r = 0, 1, 2, \dots$ This can be done thanks to the fact that only n-argument functions from $\Omega^{(r)}$ are used for the generation of the n-argument functions in $\Omega^{(r+1)}$. To find $^{s}D_{\Omega}(h)$ for a given n-argument Boolean function h, it is sufficient to carry out this process until one reaches for the first time a set $\Omega^{(r)}$ containing h as an element. The weak depth of an n-argument Boolean function with respect to a finite weakly complete set of Boolean functions can be found in a similar way.

Although the number $^{s}D_{\Omega}(h)$ depends both on the function h and on the set Ω, this number remains bounded for any fixed h. In fact, the inequality

$$^{s}D_{\Omega}(h) < 2^{2^n} \tag{2.2}$$

holds for any strongly complete set Ω of Boolean functions and any n-argument Boolean function h. To see this, suppose Ω is a strongly complete set of Boolean functions and n is a positive integer. Since there are only 2^{2^n} n-argument Boolean functions, $\Omega^{(0)}$ contains at least one of them, and $\Omega^{(r)}$ is a subset of $\Omega^{(r+1)}$ for any natural number r, it is clear that $\Omega^{(r+1)} \setminus \Omega^{(r)}$ contains no n-argument Boolean function for some r less than 2^{2^n}. Obviously, all n-argument Boolean functions will belong to $\Omega^{(r)}$ for such an r.[3]

The fact we just indicated allows us to give the following definition: for any Boolean function h, the largest of the numbers ${}^sD_\Omega(h)$, where Ω ranges over all strongly complete sets of Boolean functions, will be called *the maximal strong depth of* h and will be denoted by ${}^sD(h)$.

A quite similar reasoning shows that also ${}^wD_\Omega(h)$ remains bounded for any fixed Boolean function h when Ω ranges over all weakly complete sets of Boolean functions. For any Boolean function h, the largest of the corresponding numbers ${}^wD_\Omega(h)$ will be called *the maximal weak depth of* h and will be denoted by ${}^wD(h)$.

Example 2.1. The maximal strong depth of the negation function is equal to 2. In fact, let $h = \lambda x.\overline{x}$. If Ω is an arbitrary set of Boolean functions, then $\lambda x.x$ is the only one-argument function in $\Omega^{(0)}$. Suppose Ω consists of the constant 1, addition modulo 2 and conjunction. Then Ω is strongly complete, and the only one-argument functions in $\Omega^{(1)} \setminus \Omega^{(0)}$ are the two constants, hence ${}^sD_\Omega(h) \geq 2$. It remains to prove that h has a strong depth not greater than 2 with respect to any strongly complete set of Boolean functions. To prove this, suppose that Ω is an arbitrary strongly complete set of Boolean functions. By the Post Theorem, there are functions f_0 and f_1 in Ω such that $f_0(0,\ldots,0) = 1$ and $f_1(1,\ldots,1) = 0$. The one-argument functions

$$h_0 = \lambda x.f_0(x,\ldots,x), \quad h_1 = \lambda x.f_1(x,\ldots,x)$$

belong to $\Omega^{(1)}$. Either some of them coincides with h or these functions are the two constants. In the first case h belongs to $\Omega^{(1)}$, hence ${}^sD_\Omega(h) = 1$. In the second one we may consider some function f in Ω that is not monotonically increasing (such a function exists again by the Post Theorem). Then h can be obtained from f by substitution of constants for all its arguments except for one of them. Therefore h belongs to $\Omega^{(2)}$, hence ${}^sD_\Omega(h) \leq 2$.

Example 2.2. The maximal weak depth of the negation function is 1. Indeed, let h be again this function, and Ω be an arbitrary weakly complete set of Boolean functions. Of course, h does not belong to $\Omega^{[0]}$. Since there is a function in Ω that is not monotonically increasing, and h can be obtained from it by substitution of

[3]The set $\Omega^{(0)}$ contains in fact n different n-argument Boolean functions, therefore the above reasoning actually proves the inequality ${}^sD_\Omega(h) \leq 2^{2^n} - n$, which is stronger than (2.2) for $n > 1$. This small strengthening, however, is quite immaterial, since, as Todor Tsankov noticed, the upper bound 2^{2^n} can be replaced by another one which has a much lower order of magnitude. (His reasoning makes use of the disjunctive normal form representation of the Boolean functions.)

constants for all arguments but one, h belongs to $\Omega^{[1]}$. Thus the negation function has a weak depth 1 with respect to any weakly complete set of Boolean functions.

Remark 2.2. It can be shown that ${}^{s}D(h) \geq {}^{w}D(h)$ for any Boolean function h. In fact, if we choose a weakly complete set Ω such that ${}^{w}D_{\Omega}(h) = {}^{w}D(h)$ and define the set Ω' as in Lemma 2.1, then Ω' will be strongly complete and we shall have the inequalities ${}^{s}D(h) \geq {}^{s}D_{\Omega'}(h) \geq {}^{w}D(h)$.

The definitions of maximal strong depth and maximal weak depth of a function do not provide us with algorithms for computing these depths, because there are infinitely many strongly complete and infinitely many weakly complete sets of Boolean functions. The existence of such algorithms will be shown in the rest of the paper.

3. ALGORITHMIC COMPUTABILITY OF THE MAXIMAL WEAK DEPTH

We start with the case of the weak depths, because its treatment is much easier, and we have a result in a more finished state for this case.

We shall use the following six weakly complete sets of Boolean functions:

$$\begin{aligned}
\Omega_1 &= \{\, \lambda x.\bar{x},\ \lambda xy.xy \,\}, \\
\Omega_2 &= \{\, \lambda x.\bar{x},\ \lambda xy.x \vee y \,\}, \\
\Omega_3 &= \{\, \lambda xy.x \rightarrow y \,\}, \\
\Omega_4 &= \{\, \lambda xy.\overline{xy} \,\}, \\
\Omega_5 &= \{\, \lambda xy.\overline{x \vee y} \,\}, \\
\Omega_6 &= \{\, \lambda xy.\overline{x \rightarrow y} \,\}.
\end{aligned}$$

Lemma 3.1. *For any weakly complete set Ω of Boolean functions some of the sets $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6$ is a subset of the set $\Omega^{[1]}$.*

Proof. Let Ω be a weakly complete set of Boolean functions. Some non-linear function f surely belongs to Ω, and a two-argument non-linear function g can be obtained from f by substitution of constants for all its arguments except for two of them. The function g will belong to the set $\Omega^{[1]}$ and will have the form

$$g(x, y) = xy \oplus ax \oplus by \oplus c,$$

where a, b, c belong to $\{0, 1\}$, and "$\oplus$" denotes addition modulo 2. Without a loss of generality we may assume that $a \geq b$. If $a = c = 0$, then $g(x, y) = xy$, and from here, taking into account also Example 2.2, we see that $\Omega_1 \subseteq \Omega^{[1]}$. If $a = 0,\ c = 1$, then $g(x, y) = \overline{xy}$, hence $\Omega_4 \subseteq \Omega^{[1]}$. If $a = 1,\ b = c = 0$, then $g(x, y) = \overline{x \rightarrow y}$ and therefore $\Omega_6 \subseteq \Omega^{[1]}$. If $a = 1,\ b = 0,\ c = 1$, then $g(x, y) = x \rightarrow y$, hence $\Omega_3 \subseteq \Omega^{[1]}$.

If $a = b = 1,\ c = 0$, then $g(x, y) = x \vee y$, hence $\Omega_2 \subseteq \Omega^{[1]}$. Finally, if $a = b = c = 1$, then $g(x, y) = \overline{x \vee y}$, therefore $\Omega_5 \subseteq \Omega^{[1]}$ □.

Lemma 3.2. *Let Ω be a set of Boolean functions, and let Ω' be any subset of the set $\Omega^{[1]}$. Then $\Omega'^{[r]} \subseteq \Omega^{[r]}$ for $r = 0, 1, 2, \dots$*

Proof. Induction on r □.

Corollary 3.1. *Let Ω and Ω' be weakly complete sets of Boolean functions, and let $\Omega' \subseteq \Omega^{[1]}$. Then ${}^wD_{\Omega'}(h) \geq {}^wD_{\Omega}(h)$ for any Boolean function h.*

Theorem 3.1. *For any Boolean function h we have the equality*

$$ {}^wD(h) = \max\{{}^wD_{\Omega_1}(h), {}^wD_{\Omega_2}(h), {}^wD_{\Omega_3}(h), {}^wD_{\Omega_4}(h), {}^wD_{\Omega_5}(h), {}^wD_{\Omega_6}(h)\}. \quad (3.1) $$

Proof. Let h be an arbitrary Boolean function, and let d be the right-hand side of (3.1). If Ω is any weakly complete set of Boolean functions, then, by Lemma 3.1 and Corollary 3.1, the inequality ${}^wD_{\Omega_i}(h) \geq {}^wD_{\Omega}(h)$ holds for some $i \in \{1, 2, 3, 4, 5, 6\}$, hence $d \geq {}^wD_{\Omega}(h)$. On the other hand, by the choice of d, there is a weakly complete set Ω (some of the sets $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6$) such that $d = {}^wD_{\Omega}(h)$ □.

Since, by Remark 2.1, the right-hand side of the equality (3.1) is algorithmically computable, the above theorem shows the algorithmic computability of the maximal weak depth.

4. ALGORITHMIC COMPUTABILITY OF THE MAXIMAL STRONG DEPTH

The algorithmic computability of the maximal strong depth will be shown by means of an equality similar to (3.1), namely a finite class $\mathbb{O}$ of finite strongly complete sets of Boolean functions will be indicated such that

$$ {}^sD(h) = \max\{{}^sD_{\Omega}(h) | \Omega \in \mathbb{O}\} \quad (4.1) $$

for any Boolean function h. We shall call any such class $\mathbb{O}$ *representative for ${}^sD$.*

Before we actually indicate a class that is representative for ${}^sD$, we shall give the easily provable analogs of Lemma 3.2 and Corollary 3.1 that will be used now.

Lemma 4.1. *Let Ω be a set of Boolean functions, and let Ω' be any subset of the set $\Omega^{(1)}$. Then $\Omega'^{(r)} \subseteq \Omega^{(r)}$ for $r = 0, 1, 2, \dots$*

Proof. Induction on r □.

Corollary 4.1. *Let Ω and Ω' be strongly complete sets of Boolean functions,* *and let $\Omega' \subseteq \Omega^{(1)}$. Then ${}^sD_{\Omega'}(h) \geq {}^sD_{\Omega}(h)$ for any Boolean function h.*

To show that a finite class of finite strongly complete sets is representative for ${}^sD$, it would be sufficient to ascertain that this class has a property analogous to the property stated in Lemma 3.1.

Lemma 4.2. *Let $\mathbb{O}$ be a finite class of finite strongly complete sets of Boolean functions, and let for any strongly complete set Ω of Boolean functions there be some subset of $\Omega^{(1)}$ belonging to $\mathbb{O}$. Then $\mathbb{O}$ is representative for ${}^sD$.*

Proof. We reason as in the proof of Theorem 3.1. Let h be an arbitrary Boolean function, and let d be the right-hand side of (4.1). If Ω is any strongly complete set of Boolean functions, then, by the assumption of the lemma and by Corollary 4.1, the inequality ${}^sD_{\Omega'}(h) \geq {}^sD_{\Omega}(h)$ holds for some $\Omega' \in \mathbb{O}$, hence $d \geq {}^sD_{\Omega}(h)$. On the other hand, by the choice of d, there is a strongly complete set Ω (some of the sets belonging to $\mathbb{O}$) such that $d = {}^sD_{\Omega}(h)$ $\square$.

Having in mind the above lemma, we shall aim at indicating a class $\mathbb{O}$ that satisfies the assumption of the lemma.

For any m-argument Boolean function f, any positive integer n and any sequence $k_1, k_2, \ldots, k_m$ of numbers from the set $\{1, 2, \ldots, n\}$, the n-argument Boolean function g defined by

$$g(x_1, x_2, \ldots, x_n) = f(x_{k_1}, x_{k_2}, \ldots, x_{k_m})$$

will be called a *projection instance* of f (an *n-ary projection instance* of f). Clearly, the relation of being a projection instance is reflexive and transitive. Obviously, each Boolean function has exactly one unary projection instance. We note also that, for any set Ω of Boolean functions, the set $\Omega^{(1)}$ consists of all projection functions and all projection instances of functions of Ω.

As usually, T_0, T_1, S, M and L will denote, respectively, the class of all Boolean functions preserving 0, the class of all Boolean functions preserving 1, the class of all self-dual Boolean functions, the class of all monotonically increasing ones and the class of all linear ones. We define finite sets ${T_0}^\dagger$, ${T_1}^\dagger$, $S^\dagger$, $M^\dagger$ and $L^\dagger$ of Boolean functions as follows:

- ${T_i}^\dagger$ is the set of the one-argument Boolean functions not belonging to T_i (for $i = 0, 1$);

- $S^\dagger$ is the set of the symmetric two-argument Boolean functions;

- $M^\dagger$ is the set of the three-argument Boolean functions g satisfying the conditions $g(0, 0, 1) = 1$ and $g(1, 0, 1) = 0$;

- $L^\dagger$ consists of all two-arguments functions not belonging to L and all three-argument functions g of the form

$$g(x, y, z) = xy \oplus yz \oplus xz \oplus ax \oplus by \oplus cz \oplus d$$

with coefficients a, b, c, d in $\{0, 1\}$.

Obviously, C and $C^\dagger$ have an empty intersection for $C = T_0, T_1, S, M, L$.

The existence of a class satisfying the assumption of Lemma 4.2 will follow from the next four lemmas.

Lemma 4.3. *Let C be the class T_0 or the class T_1, and f be a Boolean function not belonging to C. Then the unary projection instance of f belongs to $C^\dagger$.*

Proof. Obvious $\square$.

Lemma 4.4. *Each Boolean function not belonging to the class S has a projection instance belonging to $S^\dagger$.*

Proof. Let f be an m-argument function not belonging to S. Then there are $a_1, a_2, \ldots, a_m$ in $\{0, 1\}$ such that

$$f(a_1, a_2, \ldots, a_m) = f(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_m}).$$

We define the function g by the equality

$$g(x_1, x_2) = f(x_{k_1}, x_{k_2}, \ldots, x_{k_m}),$$

where

$$k_i = a_i + 1,\ i = 1, 2, \ldots, m.$$

Then g is a projection instance of f and the equalities

$$g(0, 1) = f(a_1, a_2, \ldots, a_m), \quad g(1, 0) = f(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_m})$$

hold, hence $g(0, 1) = g(1, 0)$ $\square$.

Lemma 4.5. *Each Boolean function not belonging to the class M has a projection instance belonging to $M^\dagger$.*

Proof. Let f be an m-argument function not belonging to M. Then there are some j in $\{1, 2, \ldots, m\}$ and some $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_m$ in $\{0, 1\}$ such that

$$f(a_1, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_m) = 1, \ \ f(a_1, \ldots, a_{j-1}, 1, a_{j+1}, \ldots, a_m) = 0.$$

We define the function g by the equality

$$g(x_1, x_2, x_3) = f(x_{k_1}, x_{k_2}, \ldots, x_{k_m}),$$

where

$$k_i = \begin{cases} 1, & \text{if } i = j, \\ a_i + 2, & \text{if } i \in \{1, \ldots, j-1, j+1, \ldots, m\}. \end{cases}$$

Then g is a projection instance of f and the equalities

$$g(x_1, 0, 1) = f(a_1, \ldots, a_{j-1}, x_1, a_{j+1}, \ldots, a_m), \quad x_1 = 0, 1,$$

hold □.

Lemma 4.6. *Each Boolean function not belonging to the class L has a projection instance belonging to $L^\dagger$.*

Proof. To prove the statement of the lemma, it is sufficient to show, for any integer m greater than 2, that each non-linear m-argument Boolean function not belonging to $L^\dagger$ has a non-linear $(m-1)$-argument projection instance.

Let f be a non-linear m-argument Boolean function not belonging to $L^\dagger$. In the case when $m = 3$, we may reason as follows. The representation of the function f as a Zhegalkin polynomial has the form

$$f(x, y, z) = axyz \oplus b_1 yz \oplus b_2 xz \oplus b_3 xy \oplus c_1 x \oplus c_2 y \oplus c_3 z \oplus d,$$

where $a, b_1, b_2, b_3, c_1, c_2, c_3, d$ are fixed elements of the set $\{0, 1\}$, at least one of the numbers a, b_1, b_2, b_3 is not zero, and if $b_1 = b_2 = b_3 = 1$, then also $a = 1$. For all x, y, z in $\{0, 1\}$ we have

$$\begin{aligned} f(x, y, y) &= (a \oplus b_2 \oplus b_3)xy \oplus c_1 x \oplus (b_1 \oplus c_2 \oplus c_3)y \oplus d, \\ f(x, y, x) &= (a \oplus b_1 \oplus b_3)xy \oplus (b_2 \oplus c_1 \oplus c_3)x \oplus c_2 y \oplus d, \\ f(x, x, z) &= (a \oplus b_1 \oplus b_2)xz \oplus (b_3 \oplus c_1 \oplus c_2)x \oplus c_3 z \oplus d. \end{aligned}$$

If we suppose that all two-argument projection instances of f are linear, then we shall have the equalities

$$a \oplus b_2 \oplus b_3 = a \oplus b_1 \oplus b_3 = a \oplus b_1 \oplus b_2 = 0,$$

but they imply the equalities $a = 0$, $b_1 = b_2 = b_3$, and this leads to a contradiction, since some of the numbers a, b_1, b_2, b_3 is not zero. Now suppose that $m > 3$. We again represent $f(x_1, x_2, \ldots, x_m)$ as a non-linear Zhegalkin polynomial. We shall show that its non-linearity will be preserved if we do an appropriate replacement of one of the variables $x_1, x_2, \ldots, x_m$ by another of them. Clearly, the new non-linear polynomial obtained in this way can be used for the definition of a non-linear $(m-1)$-argument projection instance of f.

Case 1. *The Zhegalkin polynomial representing $f(x_1, x_2, \ldots, x_m)$ contains such a non-linear term T that some two distinct variables x_i and x_j are missing in T.* Then the replacement of x_i by x_j will leave the term T unchanged, and all other terms of the polynomial will go into terms distinct from T – they will remain unchanged or will go into terms containing x_j (depending on the absence or the presence of x_i in them). Therefore the polynomial in question will go again into a non-linear Zhegalkin polynomial.

Case 2. *For any non-linear term in the Zhegalkin polynomial representing* $f(x_1, x_2, \ldots, x_m)$, *at most one of the variables* $x_1, x_2, \ldots, x_m$ *is missing in this term.* The case splits into two subcases.

Subcase 2.1. *There is a term* T *in the polynomial such that exactly one of the variables* $x_1, x_2, \ldots, x_m$ *is missing in* T. Let x_i and x_j be two distinct variables occurring in T. Then the replacement of x_i by x_j transforms T into a non-linear term T' with two missing variables, namely x_i and the variable missing in T. It is easily seen that all other terms of the polynomial (if any) will be transformed into terms distinct from T'. In fact, T' could arise only from some term with exactly one missing variable, and that term should not contain the variable missing in T. Hence the polynomial goes again into a non-linear one.

Subcase 2.2. *The term* $x_1 x_2 \ldots x_m$ *is present in the polynomial, and no other non-linear term is present in it.* In this subcase any replacement of some of the variables $x_1, x_2, \ldots, x_m$ by another of them will transform the polynomial again into a non-linear one □.

Let us define now a class $\mathbb{O}$ as follows: $\mathbb{O}$ has as its elements all sets $\{\lambda x.\overline{x}, g, h\}$, where $g \in S^\dagger$, $h \in L^\dagger$, and all sets $\{\lambda x.0, \lambda x.1, g, h\}$, where $g \in M^\dagger$, $h \in L^\dagger$. Clearly, $\mathbb{O}$ is a finite class of finite sets of Boolean functions, and, by the Post theorem, all these sets are strongly complete.

Lemma 4.7. *For any strongly complete set* Ω *of Boolean functions there is some subset of* $\Omega^{(1)}$ *belonging to* $\mathbb{O}$.

Proof. Let Ω be an arbitrary strongly complete set of Boolean functions. By the Post theorem and the preceding four lemmas, there are functions g_0, g_1, g_2, g_3, g_4 such that each of them is a projection instance of some function from Ω, hence belongs to $\Omega^{(1)}$, and the conditions $g_0 \in {T_0}^\dagger, g_1 \in {T_1}^\dagger, g_2 \in S^\dagger, g_3 \in M^\dagger, g_4 \in L^\dagger$ are satisfied. If some of the functions g_0 and g_1 is the negation function, then $\{\lambda x.\overline{x}, g_2, g_4\}$ is a subset of $\Omega^{(1)}$ belonging to $\mathbb{O}$. Otherwise, $\{\lambda x.0, \lambda x.1, g_3, g_4\}$ is such a subset. □.

Theorem 4.1. *The class* $\mathbb{O}$ *is representative for* ${}^sD$.

Proof. By the above lemma and Lemma 4.2 □.

Of course, the algorithmic computability of the strong depth is shown by the above theorem in a fully unpractical way, since the class $\mathbb{O}$ we defined is very large. The result can be considerably improved, but this will be probably done in a further publication.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: skordev@fmi.uni-sofia.bg
http://www.fmi.uni-sofia.bg/fmi/logic/skordev/