ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 96

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 96

BOUNDS ON THE VERTEX FOLKMAN NUMBER $F(4, 4; 5)$

NEDYALKO DIMOV NENOV

For a graph G the symbol $G \to (4, 4)$ means that in every 2-coloring of the vertices of G there exists a monochromatic K_4. For the vertex Folkman number

$$F(4, 4; 5) = \min\{|V(G)| : G \to (4, 4) \text{ and } K_5 \not\subset G\}$$

we show that $16 \leqq F(4, 4; 5) \leqq 35$.

Keywords: Folkman numbers, Folkman graphs
2000 MSC: 05C55

1. NOTATION

We consider only finite, non-oriented graphs, without loops and multiple edges. We call a p-clique of the graph G a set of p vertices, each two of which are adjacent. The largest positive integer p such that the graph G contains a p-clique is denoted by $\mathrm{cl}(G)$.

In this paper we shall use also the following notation:
$V(G)$ — the vertex set of the graph G;
$E(G)$ — the edge set of the graph G;
$\overline{G}$ — the complement of G;
$G[X]$, $X \subseteq V(G)$ — the subgraph of G induced by X;
$G - X$, $X \subseteq V(G)$ — the subgraph of G induced by $V(G) \setminus X$;
$N_G(v)$, $v \in V(G)$ — the set of all vertices of G adjacent to v in G;
K_n — a complete graph on n vertices;
C_n — a simple cycle on n vertices;

$\alpha(G)$ — a vertex independence number of G, i.e. $\alpha(G) = \text{cl}(\overline{G})$.

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x,y] : x \in V(G_1),\ y \in V(G_2)\}$.

Let $G_1, \ldots, G_k$ be graphs, $V(G_i) \cap V(G_j) = \emptyset$, $i \neq j$. We denote by $\bigcup\limits_{i=1}^{k} G_i$ the graph G for which $V(G) = \bigcup\limits_{i=1}^{k} V(G_i)$, $E(G) = \bigcup\limits_{i=1}^{k} E(G_i)$.

The Ramsey number $R(p,q)$ is the smallest natural number n such that for an arbitrary n-vertex graph G either $\text{cl}(G) \geqq p$ or $\alpha(G) \geqq q$.

2. VERTEX FOLKMAN NUMBERS

Definition 2.1. Let G be a graph and p, q be natural numbers. A 2-coloring

$$V(G) = V_1 \cup V_2, \quad V_1 \cap V_2 = \emptyset$$

of the vertices of G is said to be (p,q)-free if V_1 contains no p-cliques and V_2 contains no q-cliques of G. The symbol $G \to (p,q)$ means that every 2-coloring of $V(G)$ is not (p,q)-free.

Define

$$F(p,q;s) = \min\{|V(G)| : G \to (p,q) \text{ and } \text{cl}(G) < s\}.$$

Clearly, $G \to (p,q) \ \Rightarrow\ \text{cl}(G) \geqq \max\{p,q\}$. Folkman [1] has proved that there exists a graph G such that $G \to (p,q)$ and $\text{cl}(G) = \max\{p,q\}$. Therefore

$$F(p,q;s) \text{ exist} \quad \Longleftrightarrow \quad s > \max\{p,q\} \tag{1}$$

and they are called vertex Folkman numbers.

Obviously, $K_{p+q-1} \to (p,q)$ and $K_{p+q-2} \nrightarrow (p,q)$. Hence

$$F(p,q;s) = p+q-1, \text{ if } s > p+q-1. \tag{2}$$

By (1), the numbers $F(p,q;p+q-1)$ exist only if $p+q-1 \geqq \max\{p,q\}+1$. For these numbers the following result is known ([3]):

$$F(p,q;p+q-1) = p+q-1+\max\{p,q\}. \tag{3}$$

For the numbers $F(p,p;p+1)$ in [4] it has been shown that

$$3p-2 \leqq F(p,p;p+1) \leqq \lfloor 2p!\,(e-1) \rfloor - 1. \tag{4}$$

In [7] it has been proved that

$$F(p,p;p+1) \leqq \lfloor p!\,e \rfloor - 2, \quad p \geqq 3. \tag{5}$$

For multicoloring vertex Folkman numbers see [9].

3. MAIN RESULT

By (1), the numbers $F(3,3;s)$ exist only if $s \geqq 4$. For these numbers it is known that

$$F(3,3;s) = \begin{cases} 5, & \text{if } s \geqq 6, \text{ according to (2);} \\ 8, & \text{if } s = 5, \text{ according to (3);} \\ 14, & \text{if } s = 4. \end{cases} \tag{6}$$

The inequality $F(3,3;4) \leqq 14$ is proved in [6] and the opposite inequality $F(3,3;4) \geqq 14$ is verified by means of computer in [10].

By (1), the numbers $F(4,4;s)$ exist only if $s \geqq 5$. It is known that

$$F(4,4;s) = \begin{cases} 7, & \text{if } s \geqq 8, \text{ according to (2);} \\ 11, & \text{if } s = 7, \text{ according to (3);} \\ 14, & \text{if } s = 6. \end{cases} \tag{7}$$

The inequality $F(4,4;6) \leqq 14$ is proved in [8] and the inequality $F(4,4;6) \geqq 14$ is proved in [5]. By (4), we have $10 \leqq F(4,4;5) \leqq 81$. From (5) it follows that $F(4,4;5) \leqq 63$.

Our main result is the following

Theorem. $16 \leqq F(4,4;5) \leqq 35$.

4. PROOF OF THE INEQUALITY $F(4,4;5) \leqq 61$

Let $V(C_7) = \{v_1, \dots, v_7\}$ and $E(C_7) = \{[v_i, v_{i+1}],\ i = 1, \dots, 6\} \cup \{[v_1, v_7]\}$. Consider the set $V_1 = \{v_2, v_3, v_6, v_7\} \subseteq V(C_7)$. Define $V_i = \sigma^{i-1}(V_1)$, $i = 1, \dots, 7$, where $\sigma(v_i) = v_{i+1}$, $i = 1, \dots, 6$, and $\sigma(v_7) = v_1$. We denote by Γ the extension of $\overline{C}_7$, constructed by adding the new vertices $u_1, \dots, u_7$, each two of which are not adjacent and such that $N_\Gamma(u_i) = V_i$, $i = 1, \dots, 7$. The graph Γ_i (Fig. 1) is a copy of Γ such that the map $v_k \to v_k^i$, $u_k \to u_k^i$, $k = 1, \dots, 7$, is an isomorphism between Γ and Γ_i.

Proposition 4.1. ([6]) *$\Gamma_i \to (3,3)$ and* $\operatorname{cl}(\Gamma_i) = 3$.

Proposition 4.2. *Let G be a graph such that $G \to (p,p)$. Let $V_1 \cup V_2$ be a $(p+1,p+1)$-free 2-coloring of the vertices of $\overline{K}_2 + G$, where $V(\overline{K}_2) = \{u, v\}$. Then $u, v \in V_1$ or $u, v \in V_2$.*

Proof. Assume the opposite, i.e. $u \in V_1$ and $v \in V_2$. Then $(V_1 \setminus u) \cup (V_2 \setminus v)$ is a (p,p)-free coloring of $V(G)$, which is a contradiction. □

Let G be a graph. The graph $K_1 + G$, where $V(K_1) = v$, is given on Fig. 2.

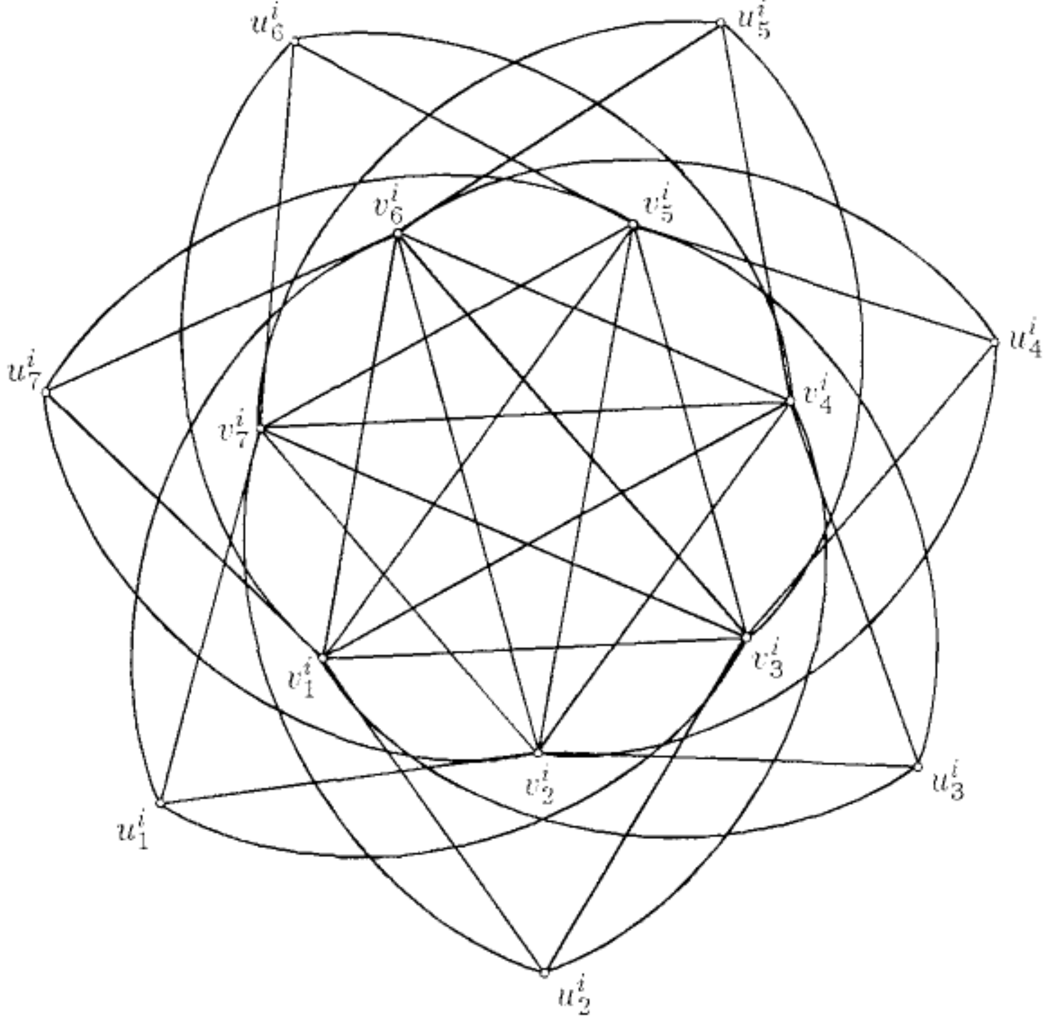

Fig. 1. The graphs Γ_i, $i = 1, 2, 3, 4$

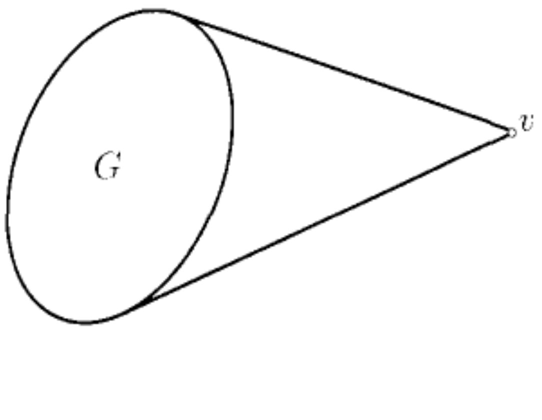

Fig. 2. The graph $K_1 + G = \{v\} + G$

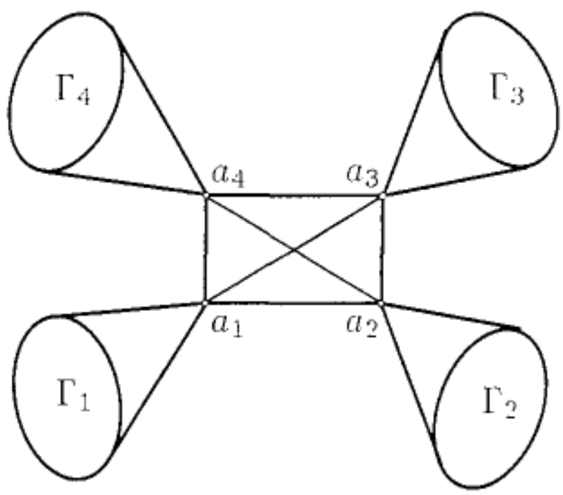

Fig. 3. The graph P

Consider the graph P with 60 vertices shown in Fig. 3, where Γ_1, Γ_2, Γ_3, Γ_4 are given in Fig. 1. We denote by Q the extension of P, constructed by adding the new vertex b such that $N_Q(b) = \bigcup_{i=1}^{4} V(\Gamma_i)$.

Proposition 4.3. $Q \to (4,4)$ *and* $\mathrm{cl}(Q) = 4$.

Proof. Since $\mathrm{cl}(\Gamma_i) = 3$, $i = 1,2,3,4$ (Proposition 4.1), we have $\mathrm{cl}(Q) = 4$. Assume that $Q \nrightarrow (4,4)$ and let $V_1 \cup V_2$ be a $(4,4)$-free 2-coloring of $V(Q)$. Without a loss of generality, we can assume that $b \in V_1$. Let $W_i = V(\Gamma_i) \cup \{a_i, b\}$. Since $Q[W_i] = \overline{K}_2 + \Gamma_i$ and $\Gamma_i \to (3,3)$, by Proposition 4.2 we have $a_i \in V_1$, $i = 1,2,3,4$. Thus V_1 contains the 4-clique $\{a_1, a_2, a_3, a_4\}$, which is a contradiction. □

Since $|V(Q)| = 61$, Proposition 4.3 implies that $F(4,4;5) \leq 61$.

5. IDENTIFICATION OF NON-ADJACENT VERTICES

Definition 5.1. Let x, y be two non-adjacent vertices in graph G. Then ${G}/{x*y}$ denote the graph G', obtained from G by identifying x and y into new vertex $x*y$, that is, $V(G') = V(G-x-y) \cup \{x*y\}$, $G' - x*y = G - x - y$ and $N_{G'}(x*y) = N_G(x) \cup N_G(y)$. Let $M' = \{x_1 * y_1, \ldots, x_{k-1} * y_{k-1}\}$ and $M = M' \cup \{x_k * y_k\}$, where $x_i, y_i \in V(G)$ and $[x_i, y_i] \notin E(G)$. Then ${G}/{M} = {G'}/{x_k * y_k}$, where $G' = {G}/{M'}$.

Proposition 5.1. *Let* $G \to (p,q)$. *Then* ${G}/{M} \to (p,q)$.

Proof. It is sufficient to prove that $G_1 = {G}/{x_1 * y_1} \to (p,q)$. Assume that $G_1 \nrightarrow (p,q)$ and let $V(G_1) = V_1 \cup V_2$ be a (p,q)-free 2-coloring. Without a lost of generality we can assume that $x_1 * y_1 \in V_1$. Let $V_1' = (V_1 \setminus \{x_1 * y_1\}) \cup \{x_1, y_1\}$. Then $V_1' \cup V_2$ is a (p,q)-free 2-coloring of $V(G)$, contradicting $G \to (p,q)$. □

Let G_1 and G_2 be isomorphic graphs without common vertices and let the map $V(G_1) \xrightarrow{\varphi} V(G_2)$ be an isomorphism. Then for $x_1, \ldots, x_k \in V(G_1)$ we define:

$$N_i = \{x_1 * \varphi(x_1), \ldots, x_i * \varphi(x_i)\}, \quad i = 1, \ldots, k;$$

$$\widetilde{G}_i = {G_1 \cup G_2}/{N_i}, \quad i = 1, \ldots, k;$$

$$V_1 = V(G_1) \setminus \{x_1, \ldots, x_k\}, \quad V_2 = V(G_2) \setminus \{\varphi(x_1), \ldots, \varphi(x_k)\};$$

$$G' = \widetilde{G}_k[V_1 \cup N_k], \quad G'' = \widetilde{G}_k[V_2 \cup N_k].$$

Proposition 5.2. $e = [x_i * \varphi(x_i), x_j * \varphi(x_j)] \in E(\widetilde{G}_k) \iff [x_i, x_j] \in E(G_1)$.

Proof. If $i = j$, Proposition 5.2 is obvious. Let $i \neq j$ and $j > i$. Clearly, $[x_i, x_j] \in E(G_1)$ implies $e \in E(\widetilde{G}_k)$. Let $[x_i, x_j] \notin E(G_1)$. Then $[\varphi(x_i), \varphi(x_j)] \notin E(G_2)$. Hence, $[x_j, x_i * \varphi(x_i)] \notin E(\widetilde{G}_i)$ and $[\varphi(x_j), x_i * \varphi(x_i)] \notin E(\widetilde{G}_i)$. Thus, $e \notin E(\widetilde{G}_j)$. From $e \notin E(\widetilde{G}_j)$ it follows $e \notin E(\widetilde{G}_k)$. □

Proposition 5.3. (a) *The graphs G' and G'' are isomorphic to the graph G_1;* (b) $\mathrm{cl}(\widetilde{G}_k) = \mathrm{cl}(G_1)$.

Proof. Define the map $\pi : V(G_1) \to V(G')$ as follows:

$$\pi(v) = v, \text{ if } v \in V_1, \quad \text{and} \quad \pi(x_i) = x_i * \varphi(x_i), \ i = 1, \dots, k.$$

Obviously, π is a bijection. By definition of G', we have

$$[u, v] \in E(G'), \ u, v \in V_1 \iff [u, v] \in E(G_1), \tag{8}$$

$$[u, x_i * \varphi(x_i)] \in E(G'), \ u \in V_1 \iff [u, x_i] \in E(G_1). \tag{9}$$

By Proposition 5.2,

$$[x_i * \varphi(x_i), x_j * \varphi(x_j)] \iff [x_i, x_j] \in E(G_1). \tag{10}$$

From (8), (9) and (10) it follows that π is an isomorphism between G_1 and G'. Similarly, it follows that G_2 and G'' are isomorphic. Since G_1 and G_2 are also isomorphic, Proposition 5.3 (a) follows. Thus, we have

$$\mathrm{cl}(G') = \mathrm{cl}(G'') = \mathrm{cl}(G_1) = \mathrm{cl}(G_2). \tag{11}$$

The proof of Proposition 5.3(b) starts by observing that

$$A \subseteq V(G') \text{ or } A \subseteq V(G'') \text{ for any clique } A \text{ of } \widetilde{G}_k. \tag{12}$$

Assume the opposite. Then there exist $u, v \in A$ such that $u \in V_1$ and $v \in V_2$. By definition of $\widetilde{G}_k$, $[u, v] \notin E(\widetilde{G}_k)$, which is a contradiction. From (12) it follows that $\mathrm{cl}(\widetilde{G}_k) = \mathrm{cl}(G')$ or $\mathrm{cl}(\widetilde{G}_k) = \mathrm{cl}(G'')$. This, together with (11), implies that $\mathrm{cl}(\widetilde{G}_k) = \mathrm{cl}(G_1)$.

6. LEMMAS

Consider the graph $L = \bigcup_{i=1}^{4} \Gamma_i$, where the graphs Γ_i are given in Fig. 1. Define:

$$M_1' = \{u_i^1 * u_i^2, \ i = 1, \dots, 7\}, \quad M_1'' = \{v_1^1 * v_1^2, \ v_2^1 * v_2^2\}, \quad M_1 = M_1' \cup M_1'';$$
$$M_2' = \{u_i^3 * u_i^4, \ i = 1, \dots, 7\}, \quad M_2'' = \{v_1^3 * v_1^4, \ v_2^3 * v_2^4\}, \quad M_2 = M_2' \cup M_2'';$$
$$M_3' = \{v_3^1 * v_3^3, \ v_4^1 * v_4^3\}, \quad M_3'' = \{v_5^1 * v_5^3, \ v_6^1 * v_6^3\}, \quad M_3 = M_3' \cup M_3'';$$
$$M_4' = \{v_3^2 * v_3^4, \ v_4^2 * v_4^4\}, \quad M_4'' = \{v_5^2 * v_5^4, \ v_6^2 * v_6^4\}, \quad M_4 = M_4' \cup M_4'';$$
$$M = \bigcup_{i=1}^{4} M_i.$$

Lemma 6.1. *The sets M_i', M_i'', $i = 1, 2, 3, 4$, are independent in graph $L/_M$.*

Proof. Observe that $\{u_1^1, \ldots, u_7^1\}$ is an independent set in Γ_1 and $\{u_1^2, \ldots, u_7^2\}$ is an independent set in Γ_2. Thus, M_1' is an independent set in ${}^{L}/_{M}$. Similarly, it follows that the other sets M_i', M_i'' are independent in ${}^{L}/_{M}$. □

Lemma 6.2. $\mathrm{cl}\left({}^{L}/_{M}\right) = 3.$

Proof. Define

$$L' = \Gamma_1 \cup \Gamma_2, \quad L'' = \Gamma_3 \cup \Gamma_4, \quad L_1 = {}^{L}/_{M_1 \cup M_2}.$$

Obviously,

$$L_1 = {}^{L'}/_{M_1} \cup {}^{L''}/_{M_2}; \tag{13}$$

$${}^{L}/_{M} = {}^{L_1}/_{M_3 \cup M_4}. \tag{14}$$

Define the map $V(\Gamma_1) \overset{\varphi}{\to} V(\Gamma_2)$ as follows:

$$v_i^1 \overset{\varphi}{\to} v_i^2, \quad u_i^1 \overset{\varphi}{\to} u_i^2, \quad i = 1, \ldots, 7.$$

Clearly, φ is an isomorphism between Γ_1 and Γ_2. Since $M_1' = \{u_i^1 * \varphi(u_i^1),\ i = 1, \ldots, 7\}$ and $M_1'' = \{v_1^1 * \varphi(v_1^1),\, v_2^1 * \varphi(v_2^1)\}$, from Proposition 4.1 and Proposition 5.3(b) it follows that

$$\mathrm{cl}\left({}^{L'}/_{M_1}\right) = 3. \tag{15}$$

Define the map $V\left({}^{L'}/_{M_1}\right) \overset{\psi}{\to} V\left({}^{L''}/_{M_2}\right)$ as follows:

$$v_i^1 \overset{\psi}{\to} v_i^3, \quad v_i^2 \overset{\psi}{\to} v_i^4, \quad i = 3, \ldots, 7;$$

$$v_1^1 * v_1^2 \overset{\psi}{\to} v_1^3 * v_1^4, \quad v_2^1 * v_2^2 \overset{\psi}{\to} v_2^3 * v_2^4;$$

$$u_i^1 * u_i^2 \overset{\psi}{\to} u_i^3 * u_i^4, \quad i = 1, \ldots, 7.$$

Obviously, ψ is an isomorphism between ${}^{L'}/_{M_1}$ and ${}^{L''}/_{M_2}$. Since

$$M_3 = \{v_i^1 * \psi(v_i^1),\ i = 3, \ldots, 6\} \quad \text{and} \quad M_4 = \{v_i^2 * \psi(v_i^2),\ i = 3, \ldots, 6\},$$

from (13) and Proposition 5.3(b) it follows that

$$\mathrm{cl}\left({}^{L_1}/_{M_3 \cup M_4}\right) = \mathrm{cl}\left({}^{L'}/_{M_1}\right). \tag{16}$$

By (14) – (16), we have $\mathrm{cl}\left({}^{L}/_{M}\right) = 3$.

7. PROOF OF THE THEOREM

I) *Proof of the inequality* $F(4,4;5) \geqq 16$. Let G be a graph such that $G \to (4,4)$ and $\mathrm{cl}(G) < 5$, i.e. $\mathrm{cl}(G) = 4$. We need to prove that $|V(G)| \geqq 16$. Observe that $|V(G)| \geqq F(4,4;6)$. Since $F(4,4;6) = 14$, [5], we have $|V(G)| \geqq 14$. From $\mathrm{cl}(G) = 4$ and $R(5,3) = 14$, [2], it follows that $\alpha(G) \geqq 3$. Let $\{v_1, v_2, v_3\}$ be an independent set in G. Then $G' = G - \{v_1, v_2, v_3\} \to (3,4)$ and $\mathrm{cl}(G') = 4$. By $F(3,4;5) = 13$, [8], we have $|V(G')| \geqq 13$. Hence, $|V(G)| \geqq 16$.

II) *Proof of the inequality* $F(4,4;5) \leqq 35$. Consider the graph $R = {}^{Q}/_{M}$, where the graph Q is defined in Section 4 and the set M is given in Section 6. Let $R_1 = R - \{a_1, a_2, a_3, a_4\}$. Observe that

$$R_1 = K_1 + {}^{L}/_{M}, \qquad \text{where } V(K_1) = \{b\} \text{ and } {}^{L}/_{M} \text{ is defined in Section 6.} \tag{17}$$

By Proposition 4.3 and Proposition 5.1, we have $R \to (4,4)$. We prove that $\mathrm{cl}(R) = 4$. Assume that $\mathrm{cl}(R) \geqq 5$ and let $A \subseteq V(R)$ be a 5-clique of R. By Lemma 6.2, $\mathrm{cl}\left({}^{L}/_{M}\right) = 3$. Since $N_R(b) = V\left({}^{L}/_{M}\right)$, this implies that $b \notin A$. From (17), $\mathrm{cl}\left({}^{L}/_{M}\right) = 3$ and $b \notin A$ it follows that $|V(R_1) \cap A| \leqq 3$. Hence,

$$|A \cap \{a_1, a_2, a_3, a_4\}| \geqq 2. \tag{18}$$

Observe that

$$N_R(a_1) \cap N_R(a_2) = M_1 \cup \{a_3, a_4\} = (M_1' \cup \{a_3\}) \cup (M_1'' \cup \{a_4\}). \tag{19}$$

By Lemma 6.1, M_1' and M_1'' are independent sets. Since $M_1' \cap N_R(a_3) = \emptyset$ and $M_1'' \cap N_R(a_4) = \emptyset$, the sets $M_1' \cup \{a_3\}$ and $M_1'' \cup \{a_4\}$ are also independent sets. Hence $M_1 \cup \{a_3, a_4\}$ contains no 3-cliques. Thus, (19) implies that $\{a_1, a_2\} \nsubseteq A$. Similarly, it follows that $\{a_i, a_j\} \nsubseteq A$, $\forall i \neq j$. This contradicts (18) and proves $\mathrm{cl}(R) = 4$. So, $R \to (4,4)$ and $\mathrm{cl}(R) = 4$. Since $|V(G)| = 35$, we have $F(4,4;5) \leqq 35$. □

REFERENCES

1. Folkman, J. Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM J. Appl. Math.*, **18**, 1970, 19–24.
2. Greenwood, R., A. Gleason. Combinatorial relation and chromatic graphs. *Canad. J. Math.*, **7**, 1955, 1–7.
3. Łuczak, T., S. Urbański. A note on restricted vertex Ramsey numbers. *Period. Math. Hungar.*, **33**, 1996, 101–103.
4. Łuczak, T., A. Ruciński, S. Urbański. On minimal vertex Folkman graph. *Discrete Math.*, **236**, 2001, 245–262.

5. Nedialkov, E., N. Nenov. Computation of the vertex Folkman number $F(4,4;6)$. In: *Proceedings of the Third Euro Workshop on Optimal Codes and related topics, Sunny Beach, Bulgaria, 11–16 June*, 2001, 123–128.

6. Nenov, N. An example of a 15-vertex (3, 3)-Ramsey graph with clique number 4. *C. R. Acad. Bulgare Sci.*, **34**, 1981, 1487–1489 (in Russian).

7. Nenov, N. Application of the corona-product of two graphs in Ramsey theory. *Ann. Univ. Sofia, Fac. Math. and Inf.*, **79**, 1985, 349–355 (in Russian).

8. Nenov, N. On the vertex Folkman number $F(3,4)$. *C. R. Acad. Bulgare Sci.*, **54**, 2001, No. 2, 23–26.

9. Nenov, N. On a class of vertex Folkman numbers. *Serdica Math. J.*, **28**, 2002, 219–232.

10. Piwakowski, K., S. Radziszowski, S. Urbański. Computation of the Folkman number $F_e(3,3;5)$. *J. Graph Theory*, **32**, 1999, 41–49.

Received on January 9, 2003

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: nenov@fmi.uni-sofia.bg
hadji@fmi.uni-sofia.bg