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DEGREE SPECTRA AND CO-SPECTRA OF STRUCTURES

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Given a countable structure $\mathfrak{A}$, we define the degree spectrum $DS(\mathfrak{A})$ of $\mathfrak{A}$ to be the set of all enumeration degrees generated by the presentations of $\mathfrak{A}$ on the natural numbers. The co-spectrum of $\mathfrak{A}$ is the set of all lower bounds of $DS(\mathfrak{A})$. We prove some general properties of the degree spectra, which show that they behave with respect to their co-spectra very much like the cones of enumeration degrees. Among the results are the analogs of Selman's Theorem [14], the Minimal Pair Theorem and the existence of a quasi-minimal enumeration degree.

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1. INTRODUCTION

Given a countable abstract structure $\mathfrak{A}$, we define the degree spectrum $DS(\mathfrak{A})$ of $\mathfrak{A}$ to be the set of all enumeration degrees generated by the presentations of $\mathfrak{A}$ on the natural numbers. The co-spectrum of $\mathfrak{A}$ is the set of all lower bounds of $DS(\mathfrak{A})$. As a typical example of a spectrum one may consider the cone of the total degrees, greater than or equal to some **a**, and the respective co-spectrum which is equal to the set of all degrees less than or equal to **a**. There are examples of structures with more complicated degree spectra, e.g. [11, 8, 2, 7, 15]. In any case the co-spectrum of a structure is a countable ideal and as we shall see, every countable ideal can be represented as co-spectrum of some structure.

Here we shall prove some general properties of the degree spectra, which show that the degree spectra behave with respect to their co-spectra very much like the cones of enumeration degrees. Among the results we would like to mention the analogs of Selman's Theorem [14], the Minimal Pair Theorem and the existence of a quasi-minimal enumeration degree. These results are known in two versions in the theory of the enumeration degrees – above one fixed degree and above a sequence of degrees, while our approach gives a unified treatment of both cases. Another possible benefit is that the objects constructed in the proofs are elements of the degree spectra or closely related to them, which gives an additional information about their complexity.

Finally, our results pose some restrictions on the sets of degrees, which can be represented as degree spectra. For example, using the existence of quasi-minimal degrees, we obtain that if a degree spectrum posseses a countable base of total degrees, then it has a least element. As a consequence of this, we get that for every two incomparable Turing degrees **a** and **b** there does not exist a structure $\mathfrak{A}$ such that $DS(\mathfrak{A})$ is equal to the union of the cones above **a** and **b**, answering negatively a question apparently posed by Goncharov.

2. PRELIMINARIES

2.1. ORDINAL NOTATIONS

In what follows we shall consider only recursive ordinals α, which are below a fixed recursive ordinal η. We shall suppose that a notation $e \in \mathcal{O}$ for η is fixed and the notations for the ordinals $\alpha < \eta$ are elements a of $\mathcal{O}$ such that $a <_o e$. For the definitions of the set $\mathcal{O}$ and the relation "$<_o$" the reader may consult [12] or [13]. We shall identify every ordinal with its notation and denote the ordinals by the letters α, β, γ and δ. In particular, we shall write $\alpha < \beta$ instead of $\alpha <_o \beta$. If α is a limit ordinal, then by $\{\alpha(p)\}_{p\in\mathbb{N}}$ we shall denote the unique strongly increasing sequence of ordinals with limit α, determined by the notation of α, and write $\alpha = \lim \alpha(p)$.

2.2. ENUMERATION DEGREES

Let A and B be sets of natural numbers. Then A is *enumeration reducible* to B, $A \leq_e B$, if $A = \Gamma_z(B)$ for some enumeration operator Γ_z. In other words, using the notation D_v for the finite set having canonical code v, and $W_0, \ldots, W_z, \ldots$ for the Gödel enumeration of the r.e. sets, we have

$$A \leq_e B \iff \exists z \forall x (x \in A \iff \exists v (\langle v, x\rangle \in W_z \ \&\ D_v \subseteq B)).$$

The relation $\leq_e$ is reflexive and transitive and induces an equivalence relation $\equiv_e$ on all subsets of $\mathbb{N}$. The respective equivalence classes are called enumeration degrees. We shall denote by $d_e(A)$ the enumeration degree containing A and by $\mathcal{D}_e = (\mathcal{D}_e, \leq, \mathbf{0}_e)$ the structure of the enumeration degrees, where " $\leq$ " is the partial ordering on $\mathcal{D}_e$, induced by " $\leq_e$ ", and $\mathbf{0}_e$ is the least enumeration degree consisting of all recursively enumerable sets. For an introduction to the enumeration degrees the reader might consult Cooper ([6]).

Given a set A of natural numbers, denote by A^+ the set $A \oplus (\mathbb{N} \setminus A)$. The set A is called *total* iff $A \equiv_e A^+$. An enumeration degree is total if it contains a total set. The substructure $\mathcal{D}_T$ of $\mathcal{D}_e$, consisting of all total degrees, is isomorphic to the structure of the Turing degrees. Therefore we may identify the Turing degrees with the total enumeration degrees.

The enumeration jump operator is defined in Cooper [5] and further studied by McEvoy [10]. Here we shall use the following definition of the e-jump, which is m-equivalent to the original one, see [10]:

Definition 2.1. Given a set A, let $K_A^0 = \{\langle x, z\rangle : x \in \Gamma_z(A)\}$. Define the *e-jump* A' of A to be the set $(K_A^0)^+$.

The following properties of the enumeration jump are proved in [10]:

Let A and B be sets of natural numbers. Set $B^{(0)} = B$ and $B^{(n+1)} = (B^{(n)})'$.

(J1) If $A \leq_e B$, then $A' \leq_e B'$.

(J2) A is Σ^0_{n+1} relatively to B iff $A \leq_e (B^+)^{(n)}$.

Given an enumeration degree $\mathbf{a} = d_e(A)$, let for every natural number n, $\mathbf{a}^{(n)} = d_e(A^{(n)})$. Notice that the jump is well defined on all enumeration degrees and that it is consistent with the Turing jump on the total enumeration degrees.

For every recursive ordinal α the α-th iteration of the enumeration jump $\mathbf{a}^{(\alpha)}$ is defined in a way similar to that one used in the definition of the α-th iteration of the Turing jump, see [17]. Again it turns out that both definitions are consistent on the total enumeration degrees.

2.3. DEGREE SPECTRA

We shall consider structures of the kind $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$, where " $=$ " and " $\neq$ " are among $R_1, \ldots, R_k$.

Enumeration of $\mathfrak{A}$ is every total surjective mapping of $\mathbb{N}$ onto $\mathbb{N}$.

Given an enumeration f of $\mathfrak{A}$ and a subset of A of $\mathbb{N}^a$, let

$$f^{-1}(A) = \{\langle x_1, \ldots, x_a\rangle : (f(x_1), \ldots, f(x_a)) \in A\}.$$

By $f^{-1}(\mathfrak{A})$ we shall denote the set $f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k)$. In particular, if $f = \lambda x.x$, then $f^{-1}(\mathfrak{A})$ will be denoted by $D(\mathfrak{A})$.

Definition 2.2. *The degree spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS(\mathfrak{A})$, then $\mathbf{a}$ is called the *degree of* $\mathfrak{A}$.

The notion of degree spectrum is introduced in [11], where the first results about degrees of structures are obtained. In [8] Knight defines the so-called jump degrees of structures:

Definition 2.3. Let $\alpha < \omega_1^{CK}$. Then the α-th jump spectrum of $\mathfrak{A}$ is the set

$$DS_\alpha = \{d_e(f^{-1}(\mathfrak{A})^{(\alpha)}) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

If $\mathbf{a}$ is the least element of DS_α, then $\mathbf{a}$ is called the *α-th jump degree of* $\mathfrak{A}$.

There are two main differences between the standard definition of the notion of degree spectrum of a structure considered in [11] and [8] and the one introduced here.

First of all, in the cited papers the pullback $f^{-1}(\mathfrak{A})$ of a structure is defined by taking into account not only the positive part of the predicates, but also the negative one. So the degree spectrum in the sense of [11] and [8] is equal to $DS(\mathfrak{A}^+)$, where

$$\mathfrak{A}^+ = (\mathbb{N}, R_1, \dots, R_k, \neg R_1, \dots, \neg R_k).$$

It can be easily seen that $DS(\mathfrak{A}^+)$ consists only of total enumeration degrees. We shall call structures of that kind *total*. More precisely,

Definition 2.4. A structure $\mathfrak{A}$ is *total* if all elements of $DS(\mathfrak{A})$ are total.

The second difference is connected to the enumerations. In [11] and [8] the degree spectra are defined by taking into account only the bijective enumerations, while we allow arbitrary surjective enumerations. The choice of the class of enumerations reflects on the notion of degree spectrum of a given structure. For example, let $\mathfrak{A} = (\mathbb{N}; =, \neq)$. Clearly, if we define the degree spectrum of $\mathfrak{A}$ by taking into account only the bijective enumerations, then it will be equal to $\{\mathbf{0}_e\}$, while if we take all surjective enumerations, then $DS(\mathfrak{A})$ will consist of all total enumeration degrees. Fortunately, this difference does not affect the notion of degree of a structure. Namely, the following Proposition is true:

Proposition 2.1. *Let f be an arbitrary enumeration of $\mathfrak{A}$. There exists a bijective enumeration g of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A}) \leq_e f^{-1}(\mathfrak{A})$.*

Proof. Let $E_f = \{\langle x, y\rangle : f(x) = f(y)\}$. Clearly, $E_f^+ \leq_e f^{-1}(\mathfrak{A})$. Define the function h by means of primitive recursion as follows:

$$\begin{array}{rcl} h(0) & \simeq & 0, \\ h(n+1) & \simeq & \mu z[(\forall k \leq n)(\langle h(k), z\rangle \notin E_f)]. \end{array}$$

Set $g(n) = f(h(n))$. Now one can easily check that g is bijective and $g^{-1}(\mathfrak{A}) \oplus E_f^+ \equiv_e f^{-1}(\mathfrak{A})$. □

The main benefit of defining $DS(\mathfrak{A})$ by taking all surjective enumerations is that it is always closed upwards with respect to the total enumeration degrees:

Proposition 2.2. *Let g be an enumeration of $\mathfrak{A}$. Suppose that F is a total set and $g^{-1}(\mathfrak{A}) \leq_e F$. There exists an enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A}) \equiv_e F$.*

Proof. Fix two distinct elements s and t of $\mathbb{N}$. Define the mapping $f(x)$ as follows:

$$f(x) \simeq \begin{cases} g(x/2), & \text{if } x \text{ is even}, \\ s, & \text{if } x = 2z+1 \text{ and } z \in F, \\ t, & \text{if } x = 2z+1 \text{ and } z \notin F. \end{cases}$$

Since "$=$" and "$\neq$" are among the underlined predicates of $\mathfrak{A}$, we have that $F \leq_e f^{-1}(\mathfrak{A})$.

To prove that $f^{-1}(\mathfrak{A}) \le_e F$, consider the predicate R_i of $\mathfrak{A}$. Let us fix two natural numbers x_s and x_t such that $g(x_s) \simeq s$ and $g(x_t) \simeq t$. Let $x_1, \dots, x_{r_i}$ be arbitrary natural numbers. Define the natural numbers $y_1, \dots, y_{r_i}$ by means of the following recursive in F procedure. Let $1 \le j \le r_i$. If x_j is even, then let $y_j x_j/2$. If $x_j = 2z+1$ and $z \in F$, then let $y_j = x_s$. If $x_j = 2z+1$ and $z \notin F$, then let $y_j = x_t$. Clearly,

$$\langle x_1, \dots, x_{r_i}\rangle \in f^{-1}(R_i) \iff \langle y_1, \dots, y_{r_i}\rangle \in g^{-1}(R_i).$$

Since $g^{-1}(\mathfrak{A}) \le_e F$, from the last equivalence it follows that $f^{-1}(R_i) \le_e F$. So we obtain that $f^{-1}(\mathfrak{A}) \le_e F$. □

Remark. The requirement that the set F is total is necessary for the truth of the proposition. Indeed, if the structure $\mathfrak{A}$ were total, then for all enumerations f of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ would be total.

The results in [11] show that there exist structures, e.g. linear orderings, which do not posses degrees. Further investigations in [8, 2, 7] show that for every recursive ordinal α there exist linear orderings with α-th jump degree $\mathbf{0}^{(\alpha)}$, which do not possess β-th jump degree for $\beta < \alpha$.

3. CO-SPECTRA OF STRUCTURES

Definition 3.1. Let $\mathcal{A}$ be a set of enumeration degrees, the *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely,

$$co(\mathcal{A}) = \{b : b \in \mathcal{D}_e \ \& \ (\forall a \in \mathcal{A})(b \le_e a)\}.$$

The co-set of the α-th jump spectrum of a structure $\mathfrak{A}$ will be called α-th jump co-spectrum of $\mathfrak{A}$ and will be denoted by $CS_\alpha(\mathfrak{A})$. In particular, if $\alpha = 0$, the set $CS_\alpha(\mathfrak{A})$ will be denoted by $CS(\mathfrak{A})$ and called co-spectrum of $\mathfrak{A}$.

Evidently, for every $\mathcal{A} \subseteq \mathcal{D}_e$ the set $co(\mathcal{A})$ is a countable ideal. As we shall see later, every countable ideal can be represented as a co-spectrum of some structure $\mathfrak{A}$.

Definition 3.2. Let $A \subseteq \mathbb{N}$, $\alpha < \omega_1^{CK}$ and let f be an enumeration of $\mathfrak{A}$. The set A is called α-*admissible* in the enumeration f if $A \le_e f^{-1}(\mathfrak{A})^{(\alpha)}$.

The set A is α-admissible in $\mathfrak{A}$ if A is admissible in all enumerations of $\mathfrak{A}$.

Clearly, an enumeration degree $\mathbf{a}$ belongs to $CS_\alpha(\mathfrak{A})$ iff $\mathbf{a}$ contains an α-admissible set. Our close goal is to show that the α-admissible sets admit a characterization in terms of the structure $\mathfrak{A}$. Thus we shall obtain some information about the elements of $CS_\alpha(\mathfrak{A})$. Our characterization is a generalization of the one presented in [3], where only total structures are considered. Another reason for presenting this characterization here is that we want to obtain an upper bound of the degrees in $DS_\alpha(\mathfrak{A})$, which determine the elements of $CS_\alpha(\mathfrak{A})$.

Let us fix a structure $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k)$.

In what follows, we shall use the term "finite part" to denote arbitrary finite mappings of $\mathbb{N}$ into $\mathbb{N}$. The finite parts will be denoted by δ, τ, ρ, etc.

Definition 3.3. Let $\alpha < \omega_1^{CK}$. An enumeration f of $\mathfrak{A}$ is α-generic if for every $\beta < \alpha$ and for every set S of finite parts such that $S \leq_e D(\mathfrak{A})^{(\beta)}$ the following condition holds:

$$(\exists \tau \subseteq f)(\tau \in S \vee (\forall \rho \supseteq \tau)(\rho \notin S)).$$

Proposition 3.1. *Suppose that $\alpha < \omega_1^{CK}$ and let f be an α-generic enumeration. Then for every $\beta < \alpha$, $f^{-1}(\mathfrak{A}) \not\leq_e D(\mathfrak{A})^{(\beta)}$ and hence $f^{-1}(\mathfrak{A})^{(\beta)} \not\leq_e D(\mathfrak{A})^{(\beta)}$.*

Proof. Let $\beta < \alpha$. Consider the set $\bar{E} = \{\langle x, y\rangle : f(x) \neq f(y)\}$. Clearly, $\bar{E} \leq_e f^{-1}(\mathfrak{A})^{(\beta)}$. Assume that $f^{-1}(\mathfrak{A}) \leq_e D(\mathfrak{A})^{(\beta)}$. Then the set

$$S = \{\tau : (\exists x, y \in Dom(\tau))(\langle x, y\rangle \in \bar{E} \;\&\; \tau(x) \simeq \tau(y))\}$$

is enumeration reducible to $D(\mathfrak{A})^{(\beta)}$ and hence there exists a $\tau \subseteq f$ such that $\tau \in S$ or $(\forall \rho \supseteq \tau)(\rho \notin S)$. Evidently, both conditions are impossible. □

Corollary 3.1. *If f is an α-generic enumeration, then $d_e(f^{-1}(\mathfrak{A})^{(\beta)})$ does not belong to $CS_\beta(\mathfrak{A})$ for any $\beta < \alpha$.*

For every α, e and x in $\mathbb{N}$ we define the relations $f \models_\alpha F_e(x)$ and $f \models_\alpha \neg F_e(x)$ as follows:

(i) $f \models_0 F_e(x)$ iff there exists a v such that $\langle v, x\rangle \in W_e$ and for all $u \in D_v$

$$(\exists i)(1 \leq i \leq k \;\&\; u = \langle i, x_1^u, \ldots, x_{r_i}^u\rangle \;\&\; (f(x_1^u), \ldots, f(x_{r_i}^u)) \in R_i);$$

(ii) Let $\alpha = \beta + 1$. Then

$$\begin{aligned} f \models_\alpha F_e(x) \iff & (\exists v)((\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(\\ & (u = \langle 0, e_u, x_u\rangle \;\&\; f \models_\beta F_{e_u}(x_u)) \vee \\ & (u = \langle 1, e_u, x_u\rangle \;\&\; f \models_\beta \neg F_{e_u}(x_u)))); \end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned} f \models_\alpha F_e(x) \iff & (\exists v)((\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(\\ & u = \langle p_u, e_u, x_u\rangle \;\&\; f \models_{\alpha(p_u)} F_{e_u}(x_u))); \end{aligned}$$

(iv) $f \models_\alpha \neg F_e(x) \iff f \not\models_\alpha F_e(x)$.

An immediate corollary of the definitions above is the following:

Lemma 3.1. *Let $A \subseteq \mathbb{N}$ and let $\alpha < \omega_1^{CK}$. Then $A \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$ iff there exists an e such that $A = \{x : f \models_\alpha F_e(x)\}$.*

For every $\alpha < \omega_1^{CK}$, e and x in $\mathbb{N}$ and every finite part τ we define the forcing relations $\tau \Vdash_\alpha F_e(x)$ and $\tau \Vdash_\alpha \neg F_e(x)$, following the definition of "$\models$":

(i) $\tau \Vdash_0 F_e(x)$ iff there exists a v such that

$$\langle v, x\rangle \in W_e \text{ and for all } u \in D_v,\ u = \langle i, x_1^u, \ldots, x_{r_i}^u\rangle, 1 \leq i \leq k,$$

$$x_1^u, \ldots, x_{r_i}^u \in \operatorname{dom}(\tau) \;\&\; (\tau(x_1^u), \ldots, \tau(x_{r_i}^u)) \in R_i;$$

(ii) Let $\alpha = \beta + 1$. Then

$$\tau \Vdash_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle 0, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta F_{e_u}(x_u)) \vee (u = \langle 1, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta \neg F_{e_u}(x_u))));$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\tau \Vdash_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(u = \langle p_u, e_u, x_u\rangle \ \&\ \tau \Vdash_{\alpha(p_u)} F_{e_u}(x_u)));$$

(iv) $\tau \Vdash_\alpha \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash_\alpha F_e(x))$.

For every recursive ordinal α, $e, x \in \mathbb{N}$ set $X^\alpha_{\langle e,x\rangle} = \{\rho : \rho \Vdash_\alpha F_e(x)\}$.

Given a sequence $\{X_n\}$ of sets of natural numbers, say that $\{X_n\}$ is *e-reducible* to the set P if there exists a recursive function g such that for all n we have $X_n = \Gamma_{g(n)}(P)$. The sequence $\{X_n\}$ is T-reducible to P if the function $\lambda n, x.\chi_{X_n}(x)$ is recursive in P.

From the definition of the enumeration jump it follows immediately that if $\{X_n\}$ is e-reducible to P, then $\{X_n\}$ is T-reducible to P'.

Lemma 3.2. *For every α the sequence $\{X^\alpha_n\}$ is uniformly in α e-reducible to $f^{-1}(\mathfrak{A})^{(\alpha)}$, and hence it is uniformly in α T-reducible to $f^{-1}(\mathfrak{A})^{(\alpha+1)}$.*

Proof. Using effective transfinite recursion and following the definition of the forcing, one can define a recursive function $g(\alpha, n)$ such that for every α, $X^\alpha_n = \Gamma_{g(\alpha,n)}(f^{-1}(\mathfrak{A})^{(\alpha)})$. □

The next properties of the forcing relation follow easily from the definitions and the previous lemma:

Lemma 3.3. (1) *Let α be a recursive ordinal, $e, x \in \mathbb{N}$, and let $\tau \subseteq \rho$ be finite parts. Then*

$$\tau \Vdash_\alpha (\neg)F_e(x) \Rightarrow \rho \Vdash_\alpha (\neg)F_e(x).$$

(2) *Let f be an α-generic enumeration. Then*

$$f \models_\alpha F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_\alpha F_e(x)).$$

(3) *Let f be an $(\alpha + 1)$-generic enumeration. Then*

$$f \models_\alpha \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_\alpha \neg F_e(x)).$$

Definition 3.4. Let $A \subseteq \mathbb{N}$ and let α be a recursive ordinal. The set A is *forcing α-definable* on $\mathfrak{A}$ if there exist a finite part δ and $e, x \in \mathbb{N}$ such that

$$A = \{x : (\exists \tau \supseteq \delta)(\tau \Vdash_\alpha F_e(x))\}.$$

Clearly, if A is forcing α-definable on $\mathfrak{A}$, then $A \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$. The vice versa is not always true. As we shall see later, the forcing α-definable sets coincide with the sets which are α-admissible in $\mathfrak{A}$.

The next proposition follows easily from the definitions:

Proposition 3.2. *Let $\mathfrak{B} = (\mathbb{N}, R'_1, \dots, R'_k)$ be a structure isomorphic to $\mathfrak{A}$ and α be a recursive ordinal. Then every forcing α-definable on $\mathfrak{B}$ set is forcing α-definable on $\mathfrak{A}$.*

Proposition 3.3. *Let α be a recursive ordinal, $\beta < \alpha$ and let $A \subseteq \mathbb{N}$ be not forcing β-definable on $\mathfrak{A}$. There exists an α-generic enumeration f of $\mathfrak{A}$ satisfying the following conditions:*

(1) $f \leq_e A^+ \oplus D(\mathfrak{A})^{(\alpha)}$;
(2) *If* $\gamma \leq \alpha$, *then* $f^{-1}(\mathfrak{A})^{(\gamma)} \leq_e f \oplus D(\mathfrak{A})^{(\gamma)}$;
(3) $A \not\leq_e f^{-1}(\mathfrak{A})^{(\beta)}$.

Proof. We shall construct the enumeration f by steps. At each step q we shall define a finite part δ_q, so that $\delta_q \subseteq \delta_{q+1}$, and take $f = \bigcup_q \delta_q$. We shall consider three kinds of steps. At steps $q = 3r$ we shall ensure that the mapping f is total and surjective. At steps $q = 3r+1$ we shall ensure that f is $(\alpha+1)$-generic and at steps $q = 3r+2$ we shall ensure that f satisfies (3).

Let S denote the set of all finite parts. If $\alpha = \xi + 1$, then for every natural number n set $Y_n = \Gamma_n(D(\mathfrak{A})^{(\xi)}) \cap S$. If $\alpha = \lim \alpha(p)$ is a limit ordinal, then set $Y_n = \Gamma_{(n)_0}(D(\mathfrak{A})^{(\alpha((n)_1))}) \cap S$.

In both cases we have that the sequence $\{Y_n\}$ is T-reducible to $D(\mathfrak{A})^{(\alpha)}$ and consists of all sets S of finite parts which are enumeration reducible to $D(\mathfrak{A})^{(\gamma)}$ for some $\gamma < \alpha$.

Let δ_0 be the empty finite part and suppose that δ_q is defined.

a) Case $q = 3r$. Let x_0 be the least natural number which does not belong to $\mathrm{dom}(\delta_q)$ and let s_0 be the least natural number which does not belong to the range of δ_q. Set $\delta_{q+1}(x_0) \simeq s_0$ and $\delta_{q+1}(x) \simeq \delta_q(x)$ for $x \neq x_0$.

b) Case $q = 3r+1$. Consider the set Y_r.

Check whether there exists an element ρ of Y_r such that $\delta_q \subseteq \rho$. If the answer is positive, then let δ_{q+1} be the least extension of δ_q belonging to Y_r. If the answer is negative, then let $\delta_{q+1} = \delta_q$.

c) Case $q = 3r+2$. Consider the set

$$C_r = \{x : (\exists \tau \supseteq \delta_q)(\tau \Vdash_\beta F_r(x))\}.$$

Clearly, C_r is forcing β-definable on $\mathfrak{A}$ and hence $C_r \neq A$. Notice that $C_r \leq_e D(\mathfrak{A})^{\beta}$ uniformly in r and δ_q. Therefore the set C_r is recursive in $D(\mathfrak{A})^{(\alpha)}$ uniformly in r and δ_q. Let x_r be the least natural number such that

$$x_r \in C_r \;\&\; x_r \notin A \vee x_r \notin C_r \;\&\; x_r \in A.$$

Suppose that $x_r \in C_r$. Then there exists a τ such that

$$\delta_q \subseteq \tau \;\&\; \tau \Vdash_\beta F_r(x_r). \tag{3.1}$$

Let δ_{q+1} be the least τ satisfying (3.1). If $x_r \notin C_r$, then set $\delta_{q+1} = \delta_q$. Notice that in this case we have $\delta_{q+1} \Vdash_\beta \neg F_r(x_r)$.

From the construction above it follows immediately that $f = \bigcup_q \delta_q$ is e-reducible to $A^+ \oplus D(\mathfrak{A})^{(\alpha)}$ and hence it satisfies (1).

Let $\gamma \leq \alpha$. Then there exists an e such that $f^{-1}(\mathfrak{A})^{(\gamma)} = \{x : f \models_\gamma F_e(x)\}$. Since f is α-generic, we can rewrite the last equality as $f^{-1}(\mathfrak{A})^{(\gamma)} = \{x : (\exists \tau \subseteq f)(\tau \Vdash_\gamma F_e(x))\}$. Therefore $f^{-1}(\mathfrak{A})^{(\gamma)} \leq_e f \oplus D(\mathfrak{A})^{(\gamma)}$.

It remains to show that $A \not\leq_e f^{-1}(\mathfrak{A})^{(\beta)}$. Towards a contradiction assume that $A \leq_e f^{-1}(\mathfrak{A})^{(\beta)}$. Then there exists an r such that

$$A = \{x : f \models_\beta F_r(x)\}.$$

Consider the step $q = 3r + 2$. By the construction we have

$$x_r \notin A \ \&\ \delta_{q+1} \Vdash_\beta F_r(x_r) \vee x_r \in A \ \&\ \delta_{q+1} \Vdash_\beta \neg F_r(x_r).$$

Hence by the genericity of f

$$x_r \notin A \ \&\ f \models_\beta F_r(x_r) \vee x_r \in A \ \&\ f \models_\beta \neg F_r(x_r).$$

A contradiction. □

Repeating the proof above without bothering about the set A, we get also the following:

Proposition 3.4. *Let α be a recursive ordinal. Then there exists an α-generic enumeration f such that f and $f^{-1}(\mathfrak{A})^{(\alpha)}$ are enumeration reducible to $D(\mathfrak{A})^{(\alpha)}$.*

Theorem 3.1. *Let α be a recursive ordinal, $\beta < \alpha$ and let $A \subseteq \mathbb{N}$ be not forcing β-definable on $\mathfrak{A}$. Let Q be a total set such that $A^+ \oplus D(\mathfrak{A})^{(\alpha)} \leq_e Q$. Then there exists an enumeration f satisfying the following conditions :*

(1) *The enumeration degree of $f^{-1}(\mathfrak{A})$ is total;*
(2) $A \not\leq_e f^{-1}(\mathfrak{A})^{(\beta)}$;
(3) $f^{-1}(\mathfrak{A})^{(\alpha)} \equiv_e Q$.

Proof. According Proposition 3.3 there exists an enumeration g of $\mathfrak{A}$ such that $g \leq_e Q$, $g^{-1}(\mathfrak{A})^{(\alpha)} \leq_e Q$ and $A \not\leq_e g^{-1}(\mathfrak{A})^{(\beta)}$.

From Jump Inversion Theorem [17] it follows that there exists a total set F such that the following assertions are true:

(i) $g^{-1}(\mathfrak{A}) \leq_e F$;
(ii) $A \not\leq_e F^{(\beta)}$;
(iv) $F^{(\alpha)} \equiv_e Q$.

By Proposition 2.2 there exists an enumeration f such that $f^{-1}(\mathfrak{A}) \equiv_e F$. □

Definition 3.5. Let Q be a total subset of $\mathbb{N}$ and $\alpha < \omega_1^{CK}$. An enumeration f of $\mathfrak{A}$ is α, Q-*acceptable* if f satisfies the following conditions:

(i) The enumeration degree of $f^{-1}(\mathfrak{A})$ is total;
(ii) $f^{-1}(\mathfrak{A})^{(\alpha)} \equiv_e Q$.

Theorem 3.2. *Let α be a recursive ordinal, $\beta < \alpha$ and let $A \subseteq \mathbb{N}$ be not forcing definable on $\mathfrak{A}$. Consider an enumeration g and a total set $Q \geq_e g^{-1}(\mathfrak{A})^{(\alpha)} \oplus A^+$. There exists an α, Q-acceptable enumeration f of $\mathfrak{A}$ such that $A \not\leq_e f^{-1}(\mathfrak{A})^{(\beta)}$.*

Proof. According Proposition 2.1 there exists a bijective enumeration h such that $h^{-1}(\mathfrak{A}) \leq_e g^{-1}(\mathfrak{A})$. Denote by $\mathfrak{B}$ the structure $(\mathbb{N}; h^{-1}(R_1), \ldots, h^{-1}(R_k))$. Clearly, A is not β-forcing definable on $\mathfrak{B}$ and $D(\mathfrak{B}) \equiv_e h^{-1}(\mathfrak{A})$. Hence $D(\mathfrak{B})^{(\alpha)} \leq_e Q$. Let i be an enumeration such that the enumeration degree of $i^{-1}(\mathfrak{B})$ is total. $i^{-1}(\mathfrak{B})^{(\alpha)} \equiv_e Q$ and $A \not\leq_e i^{-1}(\mathfrak{B})^{(\beta)}$. Set $f = \lambda x.h(i(x))$. Then $f^{-1}(\mathfrak{A}) \equiv_e i^{-1}(\mathfrak{B})$. Thus f is α, Q-acceptable and $A \not\leq_e f^{-1}(\mathfrak{A})^{(\beta)}$. □

Corollary 3.2. *For every total $Q \geq_e g^{-1}(\mathfrak{A})^{(\alpha)}$ there exists an α, Q-acceptable enumeration of $\mathfrak{A}$.*

Theorem 3.3. *Let α be a constructive ordinal and $A \subseteq \mathbb{N}$. Let $\beta < \alpha$. Consider an enumeration g of $\mathfrak{A}$. Suppose that $Q \geq_e g^{-1}(\mathfrak{A})^{(\alpha)}$, Q is a total set and for all α, Q-acceptable enumerations f of $\mathfrak{A}$ we have $A \leq_e f^{-1}(\mathfrak{A})^{(\beta)}$. Then A is forcing β-definable on $\mathfrak{A}$.*

Proof. First we shall show that $A^+ \leq_e Q$. Clearly, there exists an enumeration h of $\mathfrak{A}$ such that h is α, Q-acceptable. Then $A \leq_e h^{-1}(\mathfrak{A})^{(\beta)}$. By the monotonicity of the enumeration jump we can conclude that

$$A' \leq_e h^{-1}(\mathfrak{A})^{(\alpha)} \leq_e Q.$$

Since $A^+ \leq_e A'$, we get that $A^+ \leq_e Q$.

Assume that A is not forcing α-definable on $\mathfrak{A}$. Applying Theorem 3.2, we obtain an α, Q-acceptable enumeration f such that $A \not\leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$. A contradiction.

3.2. NORMAL FORM OF THE FORCING DEFINABLE SETS

In this subsection we shall show that the forcing definable sets on the structure $\mathfrak{A}$ coincide with the sets which are definable on $\mathfrak{A}$ by means of a certain kind of *positive* recursive Σ^0_α formulae. This formulae can be considered as a modification of the formulae introduced in [1], which is appropriate for their use on abstract structures.

Let $\mathfrak{L} = \{T_1, \ldots, T_k\}$ be the first order language corresponding to the structure $\mathfrak{A}$. So, every T_i is an r_i-ary predicate symbol. We shall suppose also fixed a sequence $\mathbb{X}_0, \ldots, \mathbb{X}_n, \ldots$ of variables. The variables will be denoted by the letters X, Y, W, possibly indexed.

Next we define for $\alpha < \omega_1^{CK}$ the Σ^+_α formulae. The definition is by transfinite recursion on α and goes along with the definition of indices (codes) for every formula. We shall leave to the reader the explicit definition of the indices of our formulae, which can be done in a natural way.

Definition 3.6.

(i) Let $\alpha = 0$. The elementary Σ^+_α formulae are formulae in prenex normal form with a finite number of existential quantifiers and a matrix which is a finite conjunction of atomic predicates built up from the variables and the predicate symbols $T_1, \ldots, T_k$.

(ii) Let $\alpha = \beta + 1$. An elementary Σ^+_α formula is in the form

$$\exists Y_1 \ldots \exists Y_m M(X_1, \ldots, X_l, Y_1, \ldots, Y_m),$$

where M is a finite conjunction of atoms of Σ^+_β formulae and negations of Σ^+_β formulae with free variables among $X_1, \ldots, X_l$, $Y_1, \ldots, Y_m$.

(iii) Let $\alpha = \lim \alpha(p)$ be a limit ordinal. The elementary Σ^+_α formulae are in the form

$$\exists Y_1 \ldots \exists Y_m M(X_1, \ldots, X_l, Y_1, \ldots, Y_m),$$

where M is a finite conjunction of $\Sigma^+_{\alpha(p)}$ formulae with free variables among $X_1, \ldots, X_l, Y_1, \ldots, Y_m$.

(iv) A Σ^+_α formula with free variables among $X_1, \ldots, X_l$ is an r.e. infinitary disjunction of elementary Σ^+_α formulae with free variables among $X_1, \ldots, X_l$.

Notice that the Σ^+_α formulae are effectively closed under existential quantification and infinitary r.e. disjunctions.

Let Φ be a Σ^+_α formula with free variables among $W_1, \ldots, W_n$ and let $t_1, \ldots, t_n$ be elements of $\mathbb{N}$. Then by $\mathfrak{A} \models \Phi(W_1/t_1, \ldots, W_n/t_n)$ we shall denote that Φ is true on $\mathfrak{A}$ under the variable assignment v such that $v(W_1) = t_1, \ldots, v(W_n) = t_n$.

Definition 3.7. Let $A \subseteq \mathbb{N}$ and let α be a constructive ordinal. The set A is *formally α-definable* on $\mathfrak{A}$ if there exists a recursive function $g(x)$ taking values indeces of Σ^+_α formulae $\Phi_{g(x)}$ with free variables among $W_1, \ldots, W_r$ and elements $t_1, \ldots, t_r$ of $\mathbb{N}$ such that for every element x of $\mathbb{N}$ the following equivalence holds:

$$x \in A \iff \mathfrak{A} \models \Phi_{g(x)}(W_1/t_1, \ldots, W_r/t_r).$$

We shall show that every forcing α-definable set is formally α-definable.

Let *var* be an effective mapping of the natural numbers onto the variables. Given a natural number x, by X we shall denote the variable $var(x)$.

Let $y_1 < y_2 < \ldots < y_k$ be the elements of a finite set D, let Q be one of the quantifiers $\exists$ or $\forall$ and let Φ be an arbitrary formula. Then by $Q(y : y \in D)\Phi$ we shall denote the formula $QY_1 \ldots QY_k\Phi$.

Lemma 3.4. *Let $D = \{w_1, \ldots, w_r\}$ be a finite and not empty set of natural numbers and x, e be elements of $\mathbb{N}$. Let $\alpha < \omega_1^{CK}$. There exists an uniform effective way to construct a Σ^+_α formula $\Phi^\alpha_{D,e,x}$ with free variables among $W_1, \ldots, W_r$ such that for every finite part δ with $\mathrm{dom}(\delta) = D$ the following equivalence is true:*

$$\mathfrak{A} \models \Phi^\alpha_{D,e,x}(W_1/\delta(w_1), \ldots, W_r/\delta(w_r)) \iff \delta \Vdash_\alpha F_e(x).$$

Proof. We shall construct the formula $\Phi^\alpha_{D,e,x}$ by means of effective transfinite recursion on α following the definition of the forcing relation "$\Vdash$".

1) Let $\alpha = 0$. Let $V = \{v : \langle v, x\rangle \in W_e\}$. Consider an element v of V. For every $u \in D_v$ define the atom Π_u as follows:

a) If $u = \langle i, x^u_1, \ldots, x^u_{r_i}\rangle$, where $1 \le i \le k$ and all $x^u_1, \ldots, x^u_{r_i}$ are elements of D, then let $\Pi_u = T_i(X^u_1, \ldots, X^u_{r_i})$;

b) Let $\Pi_u = W_1 \neq W_1$ in the other cases.

Set $\Pi_v = \bigwedge_{u \in D_v} \Pi_u$ and $\Phi^{\alpha}_{D,e,x} = \bigvee_{v \in V} \Pi_v$.

2) Let $\alpha = \beta + 1$. Let again $V = \{v : \langle v, x \rangle \in W_e\}$ and $v \in V$. For every $u \in D_v$ define the formula Π_u as follows:

a) If $u = \langle 0, e_u, x_u \rangle$, then let $\Pi_u = \Phi^{\beta}_{D,e_u,x_u}$;

b) If $u = \langle 1, e_u, x_u \rangle$, then let

$$\Pi_u = \neg[\bigvee_{D^* \supseteq D} (\exists y \in D^* \setminus D)\Phi^{\beta}_{D^*,e_u,x_u}];$$

c) Let $\Pi_u = \Phi^{\beta}_{\{0\},0,0} \wedge \neg\Phi^{\beta}_{\{0\},0,0}$ in the other cases.

Now let $\Pi_v = \bigwedge_{u \in D_v} \Pi_u$ and set $\Phi^{\alpha}_{D,e,x} = \bigvee_{v \in V} \Pi_v$.

3) Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Let $V = \{v : \langle v, x \rangle \in W_e\}$. Consider a $v \in V$. For every element $u = \langle p_u, e_u, x_u \rangle$ of D_v set $\Pi_u = \Phi^{\alpha(p_u)}_{D,e_u,x_u}$.

Set $\Pi_v = \bigwedge_{u \in D_v} \Pi_u$ and $\Phi^{\alpha}_{D,e,x} = \bigvee_{v \in V} \Pi_v$.

An easy transfinite induction on α shows that for every α the Σ^+_α formula $\Phi^{\alpha}_{D,e,x}$ satisfies the requirements of the lemma. □

Theorem 3.4. *Let $\alpha < \omega_1^{CK}$ and let $A \subseteq \mathbb{N}$ be forcing α-definable on $\mathfrak{A}$. Then A is formally α-definable on $\mathfrak{A}$.*

Proof. Suppose that for all $x \in \mathbb{N}$ we have

$$x \in A \iff (\exists \tau \supseteq \delta)(\tau \Vdash_\alpha F_e(x)).$$

Let $D = \mathrm{dom}(\delta) = \{w_1, \dots, w_r\}$ and let $\delta(w_i) = t_i$, $i = 1, \dots, r$. Consider a finite set $D^* \supseteq D$. By the previous lemma

$$\mathfrak{A} \models \exists(y \in D^* \setminus D)\Phi^{\alpha}_{D^*,e,x}(W_1/t_1, \dots, W_r/t_r)$$

if and only if there exists a finite part τ such that $\mathrm{dom}(\tau) = D^*$, $\tau \supseteq \delta$ and $\tau \Vdash_\alpha F_e(x)$.

Hence we have that for all $x \in \mathbb{N}$ the following equivalence is true:

$$x \in A \iff \mathfrak{A} \models \bigvee_{D^* \supseteq D} \exists(y \in D^* \setminus D)\Phi^{\alpha}_{D^*,e,x}(W_1/t_1, \dots, W_r/t_r).$$

Set

$$\Phi_{g(x)} = \bigvee_{D^* \supseteq D} \exists(y \in D^* \setminus D)\Phi^{\alpha}_{D^*,e,x}(W_1, \dots, W_r).$$

Clearly, for all $x \in \mathbb{N}$ we have

$$x \in A \iff \mathfrak{A} \models \Phi_{g(x)}.$$

Hence A is formally α-definable on $\mathfrak{A}$. □

Evidently, every formally α-definable set is α-admissible in all enumerations f of $\mathfrak{A}$. So we have the following theorem:

Theorem 3.5. *Let $A \subseteq \mathbb{N}$ and $\mathbf{a} = d_e(A)$. Let α be a recursive ordinal. Then the following are equivalent:*

(1) $\mathbf{a} \in CS_\alpha(\mathfrak{A})$;

(2) A *is forcing* α*-definable*;
(3) A *is formally* α*-definable*;
(4) A *is* α*-admissible in all enumerations of* $\mathfrak{A}$.

3.3. Representing the countable ideals as co-spectra of structures

In this subsection we are going to prove that every countable ideal of enumeration degrees can be represented as a co-spectrum of some structure.

Definition 3.8. Let $\mathfrak{A}$ be a countable structure. The enumeration degree $\mathbf{d}$ is called *co-degree* of $\mathfrak{A}$ if $\mathbf{d}$ is the greatest element of $CS(\mathfrak{A})$. If $\alpha < \omega_1^{CK}$ and $\mathbf{d}$ is the greatest element of CS_α, then $\mathbf{d}$ is called the α-th jump co-degree of $\mathfrak{A}$.

Clearly, if $\mathbf{d}$ is the α-th jump degree of a structure $\mathfrak{A}$, then $\mathbf{d}$ is also the α-th jump co-degree of $\mathfrak{A}$. The vice-versa is not always true. For example, let $\mathfrak{A} = (\mathbb{N}; <, =, \neq)$ be a linear ordering. It is easy to see by a direct analysis of the formally 0-definable on $\mathfrak{A}$ sets that the co-degree of $\mathfrak{A}$ is $\mathbf{0}$. On the other hand, there exist linear orderings without a degree, see [11]. From the results in [8] it follows that the first jump co-degree of $\mathfrak{A}$ is $\mathbf{0}'$ and again there are examples of linear orderings without first jump degree.

Obviously, if a structure $\mathfrak{A}$ has a co-degree, then $CS(\mathfrak{A})$ is a principle ideal. Building on results of Coles, Downey and Slaman [4], we shall show that every principle ideal of enumeration degrees can be represented as $CS(G)$ from some subgroup G of the additive group of the rational numbers $Q = (Q; +, =, \neq)$.

Let us fix a non-trivial group $G \subseteq Q$. Let $a \neq 0$ be an element of G. For every prime number p set

$$h_p(a) = \begin{cases} k, & \text{if } k \text{ is the greatest number such that } p^k | a \text{ in } G, \\ \infty, & \text{if } p^k | a \text{ in } G \text{ for all } k. \end{cases}$$

Let $p_0, p_1, \ldots$ be the standard enumeration of the prime numbers and set

$$S_a(G) = \{\langle i, j\rangle : j \leq h_{p_i}(a)\}.$$

It can be easily seen that if a and b are non-zero elements of G, then $S_a(G) \equiv_e S_b(G)$. Let $\mathbf{d}_G = d_e(S_a(G))$, where a is some non-zero element of G.

In [4] it is proved that for every total enumeration degree $\mathbf{a}$ there exists a bijective enumeration f of G such that $f^{-1}(\mathfrak{A}) \in \mathbf{a}$ if and only if $\mathbf{d}_G \leq \mathbf{a}$. Since for every enumeration f we have that $f^{-1}(G)$ is a total set and $\mathbf{d}_G \leq d_e(f^{-1}(G))$, we get the following proposition:

Proposition 3.5. $DS(G) = \{\mathbf{a} : \mathbf{a}$ *is total* $\&\ \mathbf{a} \geq \mathbf{d}_G\}$.

Corollary 3.3. $CS(G) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{d}_G\}$.

Proof. Clearly, $\mathbf{b} \in CS(G)$ if and only if for all total $\mathbf{a} \geq \mathbf{d}_G$, $\mathbf{a} \geq \mathbf{b}$. According Selman's Theorem [14] the last is equivalent to $\mathbf{d}_G \geq \mathbf{b}$. □

Corollary 3.4. *The group* G *has a degree if and only if* $\mathbf{d}_G$ *is total.*

Corollary 3.5. ([4]) *Every group $G \subseteq Q$ has a first jump degree.*

Proof. It is sufficient to show that $\mathbf{d}'_G \in DS_1(G)$. Indeed, by the Jump Inversion Theorem [16] there exists a total degree $\mathbf{a} \geq \mathbf{d}_G$ such that $\mathbf{a}' = \mathbf{d}'_G$. Obviously, $\mathbf{a}' \in DS_1(G)$. □

It remains to see that for every enumeration degree $\mathbf{d}$ there exists a subgroup G of Q such that $\mathbf{d}_G = \mathbf{d}$. Indeed, let $D \subseteq \mathbb{N}$. Consider the set

$$S = \{\langle i, j\rangle : j = 0 \vee j = 1 \ \&\ i \in D\}.$$

It is evident that $S \equiv_e D$. Consider the least subgroup G of Q containing the set $\{1/p_i^j : \langle i, j\rangle \in S\}$. Then $1 \in G$ and $S_1(G) = S$. So, $\mathbf{d}_G = d_e(D)$.

Now let us turn to the representation of an arbitrary countable ideal I of enumeration degrees. Without a loss of generality we may assume that there exists a sequence $\mathbf{b}_0 \leq \mathbf{b}_1 \leq \cdots \leq \mathbf{b}_k \ldots$ of elements of I such that

$$\mathbf{a} \in I \iff (\exists k)(\mathbf{a} \leq \mathbf{b}_k).$$

For every k fix a set $B_k \in \mathbf{b_k}$.

Consider the structure $\mathfrak{A} = (\mathbb{N}; G_\varphi, \sigma, =, \neq)$, where G_φ is the graph of the total recursive function φ such that $\varphi(\langle x, y\rangle) \simeq \langle x+1, y\rangle$ and

$$\sigma = \{\langle x, y\rangle : (\exists k)(y = 2k \vee y = 2k+1 \ \&\ x \in B_k)\}.$$

Proposition 3.6. $CS(\mathfrak{A}) = I$.

Proof. To show that $I \subseteq CS(\mathfrak{A})$, it is sufficient to see that $(\forall k)(\mathbf{b}_k \in CS(\mathfrak{A}))$. Indeed, let us fix a k and let f be an enumeration. Let $f^{-1}(G_\varphi) = G^f$, $f^{-1}(\sigma) = \sigma^f$ and fix a natural number x_k such that $f(x_k) = \langle 0, 2k+1\rangle$. Then for every $x \in \mathbb{N}$ we have

$$x \in B_k \iff (\exists y_1 \ldots \exists y_x)(G^f(x_k, y_1) \ \&\ G^f(y_1, y_2) \ \&\ G^f(y_{x-1}, y_x) \ \&\ \sigma^f(y_x)).$$

Thus $B_k \leq_e f^{-1}(\mathfrak{A})$.

To prove the inverse inclusion, we shall show that if A is a formally definable on $\mathfrak{A}$ set of natural numbers, then $A \leq B_k$ for some k. Let us suppose that g is a recursive function taking values indeces of Σ_0^+ formulae $\Phi_{g(x)}$ with free variables among $W_1, \ldots, W_r$ and $t_1, \ldots, t_k$ are natural numbers such that

$$x \in A \iff \mathfrak{A} \models \Phi_{g(x)}(W_1/t_1, \ldots, W_r/t_r).$$

Without a loss of generality we may assume that every $t_i = \langle 0, l_i\rangle$, where $l_1, \ldots, l_r$ are distinct natural numbers. Assume that $l_1, \ldots, l_s$ are the odd numbers among $l_1, \ldots, l_r$ and let $l_i = 2k_i + 1$, $i = 1, \ldots, s$. Set $k = \max(k_1, \ldots, k_s)$. We shall show that $A \leq_e B_k$. Indeed, let us consider an elementary Σ_0^+ formula

$$S = \exists Y_1 \ldots \exists Y_m M(Y_1, \ldots, Y_m, W_1, \ldots, W_r),$$

where M is a finite conjunction of the atoms $L_1, \ldots, L_p$. We shall show that there exists a uniform recursive procedure, which either decides that $\mathfrak{A} \not\models S(W_1/t_1, \ldots, W_r/t_r)$ or constructs finite sets of natural numbers $E_1, \ldots, E_s$ such that

$$\mathfrak{A} \models S(W_1/t_1, \ldots, W_r/t_r) \iff E_1 \subseteq B_{k_1} \ \&\ \ldots \ \&\ E_s \subseteq B_{k_s}.$$

Substituting all atomic predicates of the form $G_\varphi(Z,T)$ by $T = \varphi(Z)$, we may assume that the predicate G_φ does not occur in S.

1. Check if all L_i are of the form $Z \neq T$ or $\sigma(\varphi^{n_i}(Z))$. If there is an L of the form $Z \neq Z$, then yield $\mathfrak{A} \not\models S(W_1/t_1, \dots, W_r/t_r)$ and go to 6. Otherwise, for $j = 1, \dots s$ set

$$E_j = \{n_i : \sigma(\varphi^{n_i}(W_j)) \in \{L_1, \dots, L_p\}\}$$

and go to 6. If not all L_i are of the form $Z \neq T$ or $\sigma(\varphi^{n_i}(Z))$, then go to 2.

2. Remove all atomic predicates $\varphi^n(W_i) = \varphi^n(W_i)$. If there exists a predicate of the form $\varphi^{n_1}(W_i) = \varphi^{n_2}(W_j)$, where $i \neq j$, then yield $\mathfrak{A} \not\models S(W_1/t_1, \dots, W_r/t_r)$ and go to 6. Otherwise go to 3.

3. Suppose that among $L_1, \dots, L_p$ there exists an atomic predicate L of the form $\varphi^{n_1}(W_j) = \varphi^{n_2}(Z)$, where $n_1 < n_2$. Then $\mathfrak{A} \not\models S(W_1/t_1, \dots, W_r/t_r)$. Go to 6. If no such L exists, go to 4.

4. Suppose that there exists an L which is of the form $\varphi^{n_1}(Z) = \varphi^{n_2}(T)$, where $Z \notin \{W_1, \dots, W_r\}$ and $n_1 \geq n_2$. Remove L from the list and replace in the remaining atomic predicates all occurences of Z by $\varphi^{(n_2-n_1)}(T)$. Go to 1. Otherwise, check if there exists an L of the form $\varphi^{n_1}(T) = \varphi^{n_2}(Z)$, replace it by $\varphi^{n_1}(Z) = \varphi^{n_2}(T)$ and go to 3. Otherwise go to 5.

5. Consider the first L of the form $\varphi^{n_1}(Z) \neq \varphi^{n_2}(T)$, where $\max(n_1, n_2) > 0$. If the variables Z and T are distinct, then replace it by $Z \neq T$. If $Z = T$, then if $n_1 = n_2$, decide that $\mathfrak{A} \not\models S(W_1/t_1, \dots, W_r/t_r)$ and go to 6. If $n_1 \neq n_2$, then remove L from the list and go to 1. If no such L exists, go to 1.

6. End of the procedure.

Using the above procedure, we may construct an enumeration operator Γ such that for all x

$$\mathfrak{A} \models \Phi_{g(x)}(W_1/t_1, \dots, W_r/t_r) \iff x \in \Gamma(B_k).$$

Thus $A \leq_e B_k$. □

4. PROPERTIES OF THE DEGREE SPECTRA

4.1. GENERAL PROPERTIES OF UPWARDS CLOSED SETS

Definition 4.1. Consider a subset $\mathcal{A}$ of $\mathcal{D}_e$. Say that $\mathcal{A}$ is *upwards closed* if for every $\mathbf{a} \in \mathcal{A}$ all total degrees greater than $\mathbf{a}$ are contained in $\mathcal{A}$.

By Proposition 2.2 every degree spectrum is an upwards closed set of degrees. In this subsection we shall prove some properties of the upwards closed sets of degrees. The next subsection contains specific properties of the degree spectra, i.e. properties which are not true for all upwards closed sets of degrees.

Let $\mathcal{A}$ be an upwards closed set of degrees.

Notice first that if $\mathcal{B} \subseteq \mathcal{A}$, then $co(\mathcal{A}) \subseteq co(\mathcal{B})$.

Proposition 4.1. *Let* $\mathcal{A}_t = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \;\&\; \mathbf{a}$ *is total*$\}$. *Then* $co(\mathcal{A}) = co(\mathcal{A}_t)$.

Proof. A simple application of Selman's Theorem [14]. Suppose that $\mathbf{b} \in co(\mathcal{A}_t)$. Towards a contradiction assume that $\mathbf{b} \notin co(\mathcal{A})$. Then there exists an element $\mathbf{c} \in \mathcal{A}$ such that $\mathbf{b} \not\leq \mathbf{c}$. By Selman's Theorem there exists a total $\mathbf{a} \geq \mathbf{c}$ such that $\mathbf{b} \not\leq \mathbf{a}$. Clearly, $\mathbf{a} \in \mathcal{A}_t$. A contradiction. □

The next property can be obtained as an application of the Jump Inversion Theorem (JIT) from [17].

Proposition 4.2. *Let* $\mathbf{b}$ *be an arbitrary enumeration degree. Let* α *be a recursive ordinal greater than* 0. *Set*

$$\mathcal{A}_{\mathbf{b},\alpha} = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \&\ \mathbf{b} \leq \mathbf{a}^{(\alpha)}\}.$$

Then $co(\mathcal{A}) = co(\mathcal{A}_{\mathbf{b},\alpha})$.

Proof. Obviously, $co(\mathcal{A}) \subseteq co(\mathcal{A}_{\mathbf{b},\alpha})$. Assume that there exists a degree $\mathbf{c} \in co(\mathcal{A}_{\mathbf{b},\alpha}) \setminus co(\mathcal{A})$. Then there exists an $\mathbf{a} \in \mathcal{A}$ such that $\mathbf{c} \not\leq \mathbf{a}$. By the JIT there exists a total degree $\mathbf{f}$ such that $\mathbf{a} \leq \mathbf{f}$, $\mathbf{b} \leq \mathbf{f}^{(\alpha)}$ and $\mathbf{c} \not\leq \mathbf{f}$. Clearly, $\mathbf{f} \in \mathcal{A}_{\mathbf{b},\alpha}$. A contradiction. □

4.2. SPECIFIC PROPERTIES OF DEGREE SPECTRA

Let us fix an abstract structure $\mathfrak{A}$.

From Proposition 4.2 it follows that the elements of an upwards closed set $\mathcal{A}$ with arbitrary high jumps determine completely the co-set of $\mathcal{A}$. The next theorem shows that the elements of the degree spectrum $DS(\mathfrak{A})$ with low jumps also determine its co-set $CS(\mathfrak{A})$.

Let $\alpha > 0$ be a constructive ordinal and $\mathbf{b} \in DS_\alpha(\mathfrak{A})$. Denote by $\mathcal{A}$ the set $\{\mathbf{a} : \mathbf{a} \in DS(\mathfrak{A}) \ \&\ \mathbf{a}^{(\alpha)} = \mathbf{b}\}$.

Theorem 4.1. $CS(\mathfrak{A}) = co(\mathcal{A})$.

Proof. It is sufficient to show that $co(\mathcal{A}) \subseteq CS(\mathfrak{A})$. Let $\mathbf{c} \in co(\mathcal{A})$ and let C be a set in $\mathbf{c}$. We shall show that C is 0-forcing definable on $\mathfrak{A}$. Evidently, there exists an enumeration g of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A})^{(\alpha)} \in \mathbf{b}$. Since $\alpha > 0$, $Q = g^{-1}(\mathfrak{A})^{(\alpha)}$ is a total set. Let f be an α, Q-acceptable enumeration. Then $d_e(f^{-1}(\mathfrak{A})) \in \mathcal{A}$ and hence $C \leq_e f^{-1}(\mathfrak{A})$. So C is 0-admissible in all α, Q-acceptable enumerations of $\mathfrak{A}$. By Theorem 3.3, C is 0-forcing definable on $\mathfrak{A}$ and hence $\mathbf{c} \in CS(\mathfrak{A})$. □

There exists upwards closed set of enumeration degrees for which Theorem 4.1 is not true. Indeed, consider two sets of A and B of natural numbers such that $B \not\leq_e A$ and $A \not\leq_e B'$. One may take an arbitrary $B \not\leq_e \emptyset$ and construct the set A as a B'-generic set such that $B \not\leq A$. Let $\mathcal{D} = \{\mathbf{a} : \mathbf{a} \geq d_e(A)\} \cup \{\mathbf{a} : \mathbf{a} \geq d_e(B)\}$. Let $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \ \&\ \mathbf{a}' = d_e(B)'\}$. Clearly, if $\mathbf{a} \geq d_e(A)$, then $\mathbf{a} \notin \mathcal{A}$. Therefore $d_e(B)$ is the least element of $\mathcal{A}$ and hence $d_e(B) \in co(\mathcal{A})$. On the other hand, $d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.

Now we turn to an analog of the Minimal Pair Theorem for the enumeration degrees.

Given a partial mapping f of $\mathbb{N}$ into $\mathbb{N}$, let $f_0 = \lambda x.f(2x)$ and $f_1 = \lambda x.f(2x+1)$.

Definition 4.2. An enumeration f is *splitting* if the functions f_0 and f_1 are enumerations, i.e. f_0 and f_1 are surjective mappings of $\mathbb{N}$ onto $\mathbb{N}$.

Obviously, if f is a splitting enumeration, then both $f_0^{-1}(\mathfrak{A})$ and $f_1^{-1}(\mathfrak{A})$ are enumeration reducible to $f^{-1}(\mathfrak{A})$.

Lemma 4.1. *Let f be an α-generic splitting enumeration of $\mathfrak{A}$. Then both f_0 and f_1 are α-generic enumerations.*

Proof. We shall show that f_0 is α-generic. The proof of the genericity of f_1 is similar. Let $\beta < \alpha$ and let S_0 be an enumeration reducible to $D(\mathfrak{A})^{(\beta)}$ set of finite parts. Denote by S the set $\{\tau : \tau_0 \in S_0\}$. $S \leq_e S_0$ and hence there exists a $\tau \subseteq f$ such that $\tau \in S \vee (\forall \rho \supseteq \tau)(\rho \notin S)$.

Clearly, $\tau_0 \subseteq f_0$ and if $\tau \in S$, then $\tau_0 \in S_0$. Suppose that $(\forall \rho \supseteq \tau)(\rho \notin S)$. Assume that there exists a $\mu \supseteq \tau_0$ such that $\mu \in S_0$. Notice that since $\mu \supseteq \tau_0$, we have that for all x if $\tau(2x) \simeq y$, then $\mu(x) \simeq y$. Let

$$\rho(x) \simeq \begin{cases} \mu(x/2), & \text{if } x \text{ is even,} \\ \tau(x), & \text{if } x \text{ is odd.} \end{cases}$$

Then $\tau \subseteq \rho$ and $\rho_0 = \mu \in S_0$. So, $\rho \in S$. A contradiction. □

Corollary 4.1. *If f is an α-generic splitting enumeration, then $d_e(f_0^{-1}(\mathfrak{A}))^{(\beta)}$ and $d_e(f_1^{-1}(\mathfrak{A}))^{(\beta)}$ do not belong to $CS_\beta(\mathfrak{A})$ for any $\beta < \alpha$.*

Proposition 4.3. *Let f be an α-generic splitting enumeration of $\mathfrak{A}$. Set $\mathbf{f_0} = d_e(f_0^{-1}(\mathfrak{A}))$ and $\mathbf{f_1} = d_e(f_1^{-1}(\mathfrak{A}))$. Then for every β such that $\beta + 1 < \alpha$,*

$$co(\{\mathbf{f_0}^{(\beta)}, \mathbf{f_1}^{(\beta)}\}) = CS_\beta(\mathfrak{A}).$$

Proof. Let $\beta + 1 < \alpha$. It is sufficient to show that if $A \leq_e f_0^{-1}(\mathfrak{A})^{(\beta)}$ and $A \leq_e f_1^{-1}(\mathfrak{A})^{(\beta)}$, then A is β-forcing definable on $\mathfrak{A}$. Indeed, suppose that there exist e_0 and e_1 such that

$$(\forall x)((x \in A \iff f_0 \models_\beta F_{e_0}(x)) \;\&\; (x \in A \iff f_1 \models_\beta F_{e_1}(x))).$$

Consider the set

$$S = \{\tau : (\exists x)(\tau_0 \Vdash_\beta F_{e_0}(x) \;\&\; \tau_1 \Vdash_\beta \neg F_{e_1}(x) \vee \tau_0 \Vdash_\beta \neg F_{e_0}(x) \;\&\; \tau_1 \Vdash_\beta F_{e_1}(x))\}.$$

Clearly, S is an enumeration reducible to $D(\mathfrak{A})^{(\beta+1)}$ and hence there exists a $\tau \subseteq f$ such that $\tau \in S$ or τ has no extensions in S. Assume that $\tau \in S$. Then for some x we have that $f_0 \models_\beta F_{e_0}(x) \;\&\; f_1 \models_\beta \neg F_{e_1}(x)$ or $f_0 \models_\beta \neg F_{e_0}(x) \;\&\; f_1 \models_\beta F_{e_1}(x)$, which is impossible. So, there exists a $\tau \subseteq f$ such that τ has no extensions in S. We shall show that

$$A = \{x : (\exists \rho \supseteq \tau_0)(\rho \Vdash_\beta F_{e_0}(x))\}.$$

Let $x \in A$. Then $f_0 \models_\beta F_{e_0}(x)$ and hence there exists a $\rho \subseteq f_0$ such that $\rho \Vdash_\beta F_{e_0}(x)$. Then $\tau_0 \subseteq f_0$ and hence we may assume that $\tau_0 \subseteq \rho$. Assume now that for some $x \notin A$ there exists a $\rho \supseteq \tau_0$ such that $\rho \Vdash_\beta F_{e_0}(x)$. Then $f_1 \not\models_\beta F_{e_1}(x)$

and hence there exists a $\mu \subseteq f_1$ such that $\mu \Vdash_\beta \neg F_{e_1}(x)$. Again we may assume that $\tau_1 \subseteq \mu$. Now let

$$\sigma(x) \simeq \begin{cases} \rho(x/2), & \text{if } x \text{ is even,} \\ \mu([x/2]), & \text{if } x \text{ is odd.} \end{cases}$$

It is easy to see that $\sigma_0 = \rho$ and $\sigma_1 = \mu$. Therefore $\tau \subseteq \sigma$ and $\sigma \in S$. A contradiction. □

Theorem 4.2. *Let $\alpha < \omega_1^{CK}$ and let $\mathbf{b} \in DS_\alpha(\mathfrak{A})$. There exist elements $\mathbf{f_0}$ and $\mathbf{f_1}$ of $DS(\mathfrak{A})$ such that:*

(1) ${\mathbf{f_0}}^{(\alpha)} \leq \mathbf{b}$ *and* ${\mathbf{f_1}}^{(\alpha)} \leq \mathbf{b}$;
(2) *If $\beta < \alpha$, then ${\mathbf{f_0}}^{(\beta)}$ and ${\mathbf{f_1}}^{(\beta)}$ do not belong to $CS_\beta(\mathfrak{A})$;*
(3) *If $\beta + 1 < \alpha$, then $co(\{{\mathbf{f_0}}^{(\beta)}, {\mathbf{f_1}}^{(\beta)}\}) = CS_\beta(\mathfrak{A})$.*

Proof. Let g be a bijective enumeration of $\mathfrak{A}$ such that $d_e(g^{-1}(\mathfrak{A})^{(\alpha)}) \leq \mathbf{b}$. Denote by $\mathfrak{B}$ the structure $(\mathbb{N}; g^{-1}(R_1), \dots, g^{-1}(R_k))$. Clearly, $D(\mathfrak{B}) \equiv_e g^{-1}(\mathfrak{B})$ and for all β we have that $DS_\beta(\mathfrak{A}) = DS_\beta(\mathfrak{B})$ and $CS_\beta(\mathfrak{A}) = CS_\beta(\mathfrak{B})$. Let f be an α-generic splitting enumeration of $\mathfrak{B}$ such that $f^{-1}(\mathfrak{B})^{(\alpha)} \leq_e D(\mathfrak{B})^{(\alpha)}$. Set $\mathbf{f_0} = d_e(f_0^{-1}(\mathfrak{B}))$ and $\mathbf{f_1} = d_e(f_1^{-1}(\mathfrak{B}))$. Obviously, $\mathbf{f_0}$ and $\mathbf{f_1}$ satisfy the conditions (1) – (3). □

Again we have that Theorem 4.2 is not true for arbitrary upwards closed sets of degrees. Indeed, consider the finite lattice L consisting of the elements $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a} \wedge \mathbf{b}$, $\mathbf{a} \wedge \mathbf{c}$, $\mathbf{b} \wedge \mathbf{c}$, $\top$, $\bot$ such that $\top$ and $\bot$ are the greatest and the least element of L, respectively, $\mathbf{a} > \mathbf{a} \wedge \mathbf{b}$, $\mathbf{a} > \mathbf{a} \wedge \mathbf{c}$, $\mathbf{b} > \mathbf{a} \wedge \mathbf{b}$, $\mathbf{b} > \mathbf{b} \wedge \mathbf{c}$, $\mathbf{c} > \mathbf{a} \wedge \mathbf{c}$ and $\mathbf{c} > \mathbf{b} \wedge \mathbf{c}$. Since every finite lattice can be embedded in the semilatice of the Turing degrees, see p. 156 of [9], the lattice L can be embedded in $(\mathcal{D}_T, \leq)$ and hence it can be embedded in $(\mathcal{D}_e, \leq)$. So we may assume that L is a substructure of $(\mathcal{D}_e, \leq)$. Let

$$\mathcal{A} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \geq \mathbf{a} \vee \mathbf{d} \geq \mathbf{b} \vee \mathbf{d} \geq \mathbf{c}\}.$$

Clearly, $\mathcal{A}$ is an upwards closed set of enumeration degrees. Assume that there exist $\mathbf{f_0}, \mathbf{f_1} \in \mathcal{A}$ such that $co(\{\mathbf{f_0}, \mathbf{f_1}\}) = co(\mathcal{A})$. Let $\mathbf{x_0}, \mathbf{x_1} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be such that $\mathbf{f_0} \geq \mathbf{x_0}$ and $\mathbf{f_1} \geq \mathbf{x_1}$. Let $\mathbf{x_2} = \min\{\mathbf{x_0}, \mathbf{x_1}\}$. Then $\mathbf{x_2} \in co(\{\mathbf{f_0}, \mathbf{f_1}\})$, but $\mathbf{x_2} \notin co(\mathcal{A})$. A contradiction.

Now we turn to the third property of $DS(\mathfrak{A})$, showing the existence of enumeration degrees, which are quasi-minimal with respect to $CS(\mathfrak{A})$.

Let $\bot \notin \mathbb{N}$.

Definition 4.3. A *partial finite part* is a finite mapping of $\mathbb{N}$ into $\mathbb{N} \cup \{\bot\}$. A *partial enumeration* is a partial surjective mapping of $\mathbb{N}$ onto $\mathbb{N}$.

From now on, by δ,ρ,τ we shall denote partial finite parts. Given a partial finite part τ and a partial enumeration f, by $\tau \subseteq f$ we shall denote that for all x in $\mathrm{dom}(\tau)$ either $\tau(x) \simeq \bot$ and $f(x)$ is not defined or $\tau(x) \in \mathbb{N}$ and $f(x) \simeq \tau(x)$.

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ be a structure and f be a partial enumeration. Given a subset A of $\mathbb{N}^a$, let

$$f^{-1}(A) = \{\langle x_1, \ldots, x_a\rangle : x_1, \ldots, x_a \in \mathrm{dom}(f) \ \&\ (f(x_1), \ldots, f(x_a)) \in A\}.$$

Let $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k)$. As we shall see later, it could happen that $d_e(f^{-1}(\mathfrak{A})) \notin DS(\mathfrak{A})$. On the other hand, next lemma shows that for every partial enumeration f the enumeration degree of $f^{-1}(\mathfrak{A})$ is "almost" in $DS(\mathfrak{A})$.

Lemma 4.2. *Let X be a total set, let f be a partial enumeration and $f^{-1}(\mathfrak{A}) \leq_e X$. Then $d_e(X) \in DS(\mathfrak{A})$.*

Proof. It is sufficient to show that there exists a total surjective mapping g of $\mathbb{N}$ onto $\mathbb{N}$ such that $g^{-1}(\mathfrak{A}) \leq_e X$. Let $E_f = f^{-1}("=")$. Clearly, $E_f \leq_e X$. Since $\mathrm{dom}(f) = \{x : \langle x, x\rangle \in E_f\}$, we get that $\mathrm{dom}(f) \leq_e X$ and hence, since X is a total set, $\mathrm{dom}(f)$ is r.e. in X. Let h be a recursive in X enumeration of $\mathrm{dom}(f)$. Set $g = \lambda n.f(h(n))$. Then for every i, $1 \leq i \leq k$, we have

$$g^{-1}(R_i) = \{\langle n_1, \ldots, n_{r_i}\rangle : \langle h(n_1), \ldots, h(n_{r_i})\rangle \in f^{-1}(R_i)\}.$$

Thus $g^{-1}(\mathfrak{A}) \leq_e X$. □

Corollary 4.2. *For every partial enumeration f the enumeration degree of $f^{-1}(\mathfrak{A})'$ belongs to $DS_1(\mathfrak{A})$.*

Proof. By the Jump Inversion Theorem from [16] there exists a total set F such that $f^{-1}(\mathfrak{A}) \leq_e F$ and $F' \equiv_e f^{-1}(\mathfrak{A})'$. Then $d_e(F) \in DS(\mathfrak{A})$ and, hence, $d_e(F') \in DS_1(\mathfrak{A})$. □

Corollary 4.3. *Let f be a partial enumeration. Then $d_e(f^{-1}(\mathfrak{A}))$ is an upper bound of $CS(\mathfrak{A})$.*

Proof. Let $\mathbf{a} \in CS(\mathfrak{A})$ and let $A \in \mathbf{a}$. Consider a total set X such that $f^{-1}(\mathfrak{A}) \leq_e X$. Then $d_e(X) \in DS(\mathfrak{A})$ and hence $A \leq_e X$. By Selman's Theorem [14], $A \leq_e f^{-1}(\mathfrak{A})$. □

Definition 4.4. Let f be a partial enumeration of $\mathfrak{A}$ and $e, x \in \mathbb{N}$. Then:

(i) $f \models_0 F_e(x)$ iff there exists a v such that $\langle v, x\rangle \in W_e$ and for all $u \in D_v$

$(\exists i)(1 \leq i \leq k \ \&\ u = \langle i, x_1^u, \ldots, x_{r_i}^u\rangle \ \&\ \{x_1^u, \ldots, x_{r_i}^u\} \subseteq \mathrm{dom}(f)\ \&$

$(f(x_1^u), \ldots, f(x_{r_i}^u)) \in R_i)$;

(ii) $f \models_0 \neg F_e(x) \iff f \not\models_0 F_e(x)$.

It is obvious that $A \leq_e f^{-1}(\mathfrak{A})$ iff there exist an e such that

$$(\forall x \in \mathbb{N})(x \in A \iff f \models_0 F_e(x)).$$

Definition 4.5. Let τ be a partial finite part and $e, x \in \mathbb{N}$. Then:

(i) $\tau \Vdash_0 F_e(x)$ iff there exists a v such that $\langle v, x\rangle \in W_e$ and for all $u \in D_v$, $u = \langle i, x_1^u, \ldots, x_{r_i}^u\rangle, 1 \leq i < k$,

$$x_1^u, \ldots, x_{r_i}^u \in \mathrm{dom}(\tau) \ \&\ (\tau(x_1^u), \ldots, \tau(x_{r_i}^u)) \in R_i;$$

(ii) $\tau \Vdash_0 \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash_0 F_e(x))$.

Definition 4.6. A subset A of $\mathbb{N}$ is *partially forcing definable* on $\mathfrak{A}$ if there exist an $e \in \mathbb{N}$ and a partial finite part δ such that for all natural numbers x,

$$x \in A \iff (\exists \tau \supseteq \delta)(\tau \Vdash_0 F_e(x)).$$

Clearly, if A is partially forcing definable on $\mathfrak{A}$, then $A \leq_e D(\mathfrak{A})$.

Lemma 4.3. *Let $A \subseteq \mathbb{N}$ be partially forcing definable on $\mathfrak{A}$. Then $d_e(A) \in CS(\mathfrak{A})$.*

Proof. Let g be an arbitrary (total) enumeration of $\mathfrak{A}$. Consider a structure $\mathfrak{B}$, which is isomorphic to $\mathfrak{A}$ and such that $D(\mathfrak{B}) \leq_e g^{-1}(\mathfrak{A})$. Then A is partially forcing definable on $\mathfrak{B}$ and hence $A \leq_e D(\mathfrak{B}) \leq_e g^{-1}(\mathfrak{A})$. □

Definition 4.7. A partial enumeration f is *generic* if for every enumeration reducible to $D(\mathfrak{A})$ set S of partial finite parts the following condition holds:

$$(\exists \tau \subseteq f)(\tau \in S \vee (\forall \rho \supseteq \tau)(\rho \notin S)).$$

We shall list some properties of the partial generic enumerations omitting the proofs, since they are similar to the proofs of the respective properties of the total generic enumerations.

Proposition 4.4. (1) *For every partial generic f, $f^{-1}(\mathfrak{A}) \nleq_e D(\mathfrak{A})$. Hence $d_e(f^{-1}(\mathfrak{A})) \notin CS(\mathfrak{A})$.*

(2) *If f is a partial generic enumeration, then*

$$(\forall e, x)(f \models_0 (\neg)F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_0 (\neg)F_e(x))).$$

(3) *There exists a partial generic enumeration $f \leq_e D(\mathfrak{A})'$ such that $f^{-1}(\mathfrak{A}) \leq_e D(\mathfrak{A})'$.*

(4) *If $A \leq_e f^{-1}(\mathfrak{A})$ for all partial generic enumerations f, then A is partially forcing definable on $\mathfrak{A}$.*

Definition 4.8. Given a set $\mathcal{A}$ of enumeration degrees, say that the degree $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$ if the following conditions hold:

(i) $\mathbf{q} \notin co(\mathcal{A})$;

(ii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$;

(iii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

Notice that from (ii) it follows by Selman's Theorem that every quasi-minimal degree is an upper bound of $co(\mathcal{A})$.

If for some $\mathbf{d} \in \mathcal{D}_e$, $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \geq \mathbf{d}\}$, then a degree is quasi-minimal with respect to $\mathcal{A}$ iff it is quasi-minimal over $\mathbf{d}$.

Theorem 4.3. *Let f be a partial generic enumeration of $\mathfrak{A}$. Then $d_e(f^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to $DS(\mathfrak{A})$.*

Proof. It is sufficient to show that if ψ is a total function and $\psi \leq_e f^{-1}(\mathfrak{A})$, then $d_e(\psi) \in CS(\mathfrak{A})$. Suppose that ψ is a total function and

$$(\forall x, y \in \mathbb{N})(\psi(x) \simeq y \iff f \models_0 F_e(\langle x, y \rangle)).$$

Consider the set

$$S_0 = \{\rho : (\exists x, y_1 \neq y_2)(\rho \Vdash_0 F_e(\langle x, y_1 \rangle) \ \&\ \rho \Vdash_0 F_e(\langle x, y_2 \rangle))\}.$$

Since $S_0 \leq_e D(\mathfrak{A})$, we have that there exists a partial finite part $\tau_0 \subseteq f$ such that either $\tau_0 \in S_0$ or $(\forall \rho \supseteq \tau_0)(\rho \notin S_0)$. Assume that $\tau_0 \in S_0$. Then there exist $x, y_1 \neq y_2$ such that $f \models_0 F_e(\langle x, y_1 \rangle)$ and $f \models_0 F_e(\langle x, y_2 \rangle)$. Then $\psi(x) \simeq y_1$ and $\psi(x) \simeq y_2$, which is impossible. So, $(\forall \rho \supseteq \tau_0)(\rho \notin S_0)$.

Let

$$\begin{aligned} S_1 = \{\rho :& (\exists \tau \supseteq \tau_0)(\exists \delta_1 \supseteq \tau)(\exists \delta_2 \supseteq \tau)(\exists x, y_1 \neq y_2)(\tau \subseteq \rho \ \&\ \delta_1 \Vdash_0 F_e(\langle x, y_1 \rangle) \ \&\\ & \delta_2 \Vdash_0 F_e(\langle x, y_2 \rangle) \ \&\ \mathrm{dom}(\rho) = \mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2) \ \&\\ & (\forall x \in \mathrm{dom}(\rho) \setminus \mathrm{dom}(\tau))(\rho(x) \simeq \bot))\}. \end{aligned}$$

Again we have $S_1 \leq_e D(\mathfrak{A})$ and hence there exists a $\tau_1 \subseteq f$ such that either $\tau_1 \in S_1$ or $(\forall \rho \supseteq \tau_1)(\rho \notin S_1)$.

Assume $\tau_1 \in S_1$. Then there exists a τ such that $\tau_0 \subseteq \tau \subseteq \tau_1$ and for some $\delta_1 \supseteq \tau$, $\delta_2 \supseteq \tau$ and $x, y_1 \neq y_2 \in \mathbb{N}$ we have

$$\delta_1 \Vdash_0 F_e(\langle x, y_1 \rangle) \ \&\ \delta_2 \Vdash_0 F_e(\langle x, y_2 \rangle) \ \&\ \mathrm{dom}(\tau_1) = \mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2) \ \&$$
$$(\forall x \in \mathrm{dom}(\tau_1) \setminus \mathrm{dom}(\tau))(\tau_1(x) \simeq \bot).$$

Let $\psi(x) \simeq y$. Then $f \models_0 F_e(\langle x, y \rangle)$. Hence there exists a $\rho \supseteq \tau_1$ such that $\rho \Vdash_0 F_e(\langle x, y \rangle)$. Let $y \neq y_1$. Define the partial finite part ρ_0 as follows:

$$\rho_0(x) \simeq \begin{cases} \delta_1(x), & \text{if } x \in \mathrm{dom}(\delta_1), \\ \rho(x), & \text{if } x \in \mathrm{dom}(\rho) \setminus \mathrm{dom}(\delta_1). \end{cases}$$

Then $\tau_0 \subseteq \rho_0$, $\delta_1 \subseteq \rho_0$ and for all $x \in \mathrm{dom}(\rho)$ if $\rho(x) \not\simeq \bot$, then $\rho(x) \simeq \rho_0(x)$. Hence $\rho_0 \Vdash_0 F_e(\langle x, y_1 \rangle)$ and $\rho_0 \Vdash_0 F_e(\langle x, y \rangle)$. So, $\rho_0 \in S_0$. A contradiction.

Thus, if $\rho \supseteq \tau_1$, then $\rho \notin S_1$.

Let $\tau = \tau_1 \cup \tau_0$. Notice that $\tau \subseteq f$. We shall show that

$$\psi(x) \simeq y \iff (\exists \delta \supseteq \tau)(\delta \Vdash_0 F_e(\langle x, y \rangle)).$$

The left to right implication is trivial. Assume that $\delta_1 \supseteq \tau$, $\delta_1 \Vdash_0 F_e(\langle x, y_1 \rangle)$, $\psi(x) \simeq y_2$ and $y_1 \neq y_2$. Then there exists a $\delta_2 \supseteq \tau$ such that $\delta_2 \Vdash_0 F_e(\langle x, y_2 \rangle)$. Set

$$\rho(x) \simeq \begin{cases} \tau(x), & \text{if } x \in \mathrm{dom}(\tau), \\ \bot, & \text{if } x \in (\mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2)) \setminus \mathrm{dom}(\tau). \end{cases}$$

Then $\rho \supseteq \tau_1$ and $\rho \in S_1$. A contradiction.

Thus ψ is partially forcing definable and hence $d_e(\psi) \in CS(\mathfrak{A})$. □

As we have already pointed out, not every structure has a degree, i.e. it is not generally true that the set $DS(\mathfrak{A})$ has a least element. The next theorem shows that if $\mathfrak{A}$ has no degree, then for every countable subset $\mathcal{B} \subseteq DS(\mathfrak{A})$ of total enumeration degrees there exists an element $\mathbf{a}$ of $DS(\mathfrak{A})$ such that $(\forall \mathbf{b} \in \mathcal{B})(\mathbf{b} \not\leq \mathbf{a})$.

Definition 4.9. Let $\mathcal{A}$ be a set of enumeration degree. The subset $\mathcal{B}$ of $\mathcal{A}$ is called *base* of $\mathcal{A}$ if for every element $\mathbf{a}$ of $\mathcal{A}$ there exists an element $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{b} \leq \mathbf{a}$.

We need the following lemma, which can be proved by a minor modification of the proof of Selman's Theorem presented in [16]:

Lemma 4.4. *Let $Q \subseteq \mathbb{N}$ and let $\{B_n\}_{n\in\omega}$ be a sequence of sets of natural numbers such that $(\forall n)(B_n \not\leq_e Q)$. Then there exists a total set F such that $Q \leq_e F$ and $(\forall n)(B_n \not\leq_e F)$.*

Theorem 4.4. *Let $\mathcal{A}$ be a set of enumeration degrees possessing a quasi-minimal degree $\mathbf{q}$. Suppose that there exists a countable base $\mathcal{B}$ of $\mathcal{A}$ consisting of total degrees. Then $\mathcal{A}$ has a least element.*

Proof. Towards a contradiction assume that for every $\mathbf{b} \in \mathcal{B}$ we have $\mathbf{b} \not\leq \mathbf{q}$. Let $Q \in \mathbf{q}$ and $\{B_n : n \in \omega\}$ be a sequence of sets such that $\mathcal{B} = \{d_e(B_n) : n \in \omega\}$. Clearly, for all n, $B_n \not\leq_e Q$. Let F be a total set such that $Q \leq_e F$ and $(\forall n)(B_n \not\leq_e F)$. Set $\mathbf{f} = d_e(F)$. Then $\mathbf{f}$ is in $\mathcal{A}$ and for every $\mathbf{b} \in \mathcal{B}$ we have $\mathbf{b} \not\leq \mathbf{f}$. A contradiction. So there exists a $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{b} \leq \mathbf{q}$. Since $\mathbf{b}$ is a total degree, $\mathbf{b} \in co(\mathcal{A})$. Therefore $\mathbf{b}$ is the least element of $\mathcal{A}$. □

Corollary 4.4. *If $DS(\mathfrak{A})$ has a countable base of total enumeration degrees, then $DS(\mathfrak{A})$ has a least element.*

Now it is easy to construct an upwards closed set $\mathcal{A}$ of degrees, which does not possess a quasi-minimal degree. Indeed, let $\mathbf{a}$ and $\mathbf{b}$ be two incomparable total degrees. Let $\mathcal{A} = \{\mathbf{c} : \mathbf{c} \geq \mathbf{a} \vee \mathbf{c} \geq \mathbf{b}\}$. Then $\mathcal{A}$ has a countable base of total degrees, but it has not a least element. So, $\mathcal{A}$ has no quasi-minimal degree.

Corollary 4.4 remains true if we consider the more restrictive definition of $DS(\mathfrak{A})$, which takes into account only the bijective enumerations of $\mathfrak{A}$. Let

$$\overline{DS}(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \text{ is a bijective enumeration of } \mathfrak{A}\}.$$

Corollary 4.5. *Let $\overline{DS}(\mathfrak{A})$ have a countable base $\mathcal{B}$. Then $\overline{DS}(\mathfrak{A})$ has a least element.*

Proof. According Proposition 2.1, if $DS(\mathfrak{A})$ has a least element $\mathbf{b}$, then $\mathbf{b}$ will be the least element of $\overline{DS}(\mathfrak{A})$. So, it is sufficient to show that $DS(\mathfrak{A})$ has a least element. We shall show that $\mathcal{B}$ is a base of $DS(\mathfrak{A})$. Indeed, consider an element $\mathbf{a}$ of $DS(\mathfrak{A})$. By Proposition 2.1, there exists a $\mathbf{d} \in \overline{DS}(\mathfrak{A})$ such that $\mathbf{d} \leq \mathbf{a}$. Clearly, there exists a $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{b} \leq \mathbf{d} \leq \mathbf{a}$. □

Finally, we would like to point out the following application of the existence of a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.

Definition 4.10. The structure $\mathfrak{B}$ is called quasi-minimal with respect to $\mathfrak{A}$ if the following are true:

(i) $DS(\mathfrak{B}) \subseteq DS(\mathfrak{A})$;

(ii) $CS(\mathfrak{A}) \neq CS(\mathfrak{B})$;

(iii) If $\mathbf{a}$ is a total degree in $CS(\mathfrak{B})$, then $\mathbf{a} \in CS(\mathfrak{A})$.

Theorem 4.5. *There exists a quasi-minimal with respect to $\mathfrak{A}$ structure.*

Proof. Let $\mathbf{q}$ be a quasi-minimal with respect to $DS(\mathfrak{A})$ degree. Consider a subgroup G of the group of the rational numbers such that

$$DS(G) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{q} \leq \mathbf{a}\}.$$

Now (i) is obvious, (ii) follows from the fact that $\mathbf{q} \in CS(G)$, but $\mathbf{q} \notin CS(\mathfrak{A})$, and (iii) follows from the quasi-minimality of $\mathbf{q}$. □

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