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A DOLBEAULT ISOMORPHISM FOR COMPLETE INTERSECTIONS IN INFINITE-DIMENSIONAL PROJECTIVE SPACE

BORIS KOTZEV

We consider a complex submanifold X of finite codimension in an infinite-dimensional complex projective space $\boldsymbol{P}$ and a holomorphic vector bundle E over X. Given a covering $\mathcal{U}$ of X with Zariski open sets, we define a subcomplex $\mathcal{C}(X,E)$ of the Čech complex corresponding to the vector bundle E and the covering $\mathcal{U}$. For a special class of coverings $\mathcal{U}$, we prove that the complex $\mathcal{C}(X,E)$ is acyclic when X is a complete intersection and $\boldsymbol{P}$ admits smooth partitions of unity.

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1. INTRODUCTION

In finite dimensions the Čech cohomology groups and the Dolbeault cohomology groups of a vector bundle over a complex manifold are the same, by the Dolbeault isomorphism. When we try to extend the Dolbeault isomorphism to complex manifolds modeled on infinite-dimensional complex Banach spaces, we encounter a serious obstacle: the existence of Banach spaces for which the Dolbeault lemma about the local solvability of the $\overline{\partial}$-equation is no longer true (see [7]). In this paper we offer a way to overcome this obstacle for a projective space $\boldsymbol{P}(V)$ where V is an arbitrary Banach space. Given a holomorphic vector bundle $E \to \boldsymbol{P}(V)$ and a covering $\mathcal{U} = \{U_i\}_{i\in I}$ of $\boldsymbol{P}(V)$ with Zariski open sets, we define a subcomplex

$\mathcal{C}(\boldsymbol{P}(V), E)$ of the Čech complex corresponding to E and $\mathcal{U}$. We show in Theorem 5.1 that if $\dim \boldsymbol{P}(V) = \infty$ and $\boldsymbol{P}(V)$ admits smooth partitions of unity, then the cohomology groups $H^q(\mathcal{C}(\boldsymbol{P}(V), E))$, $q \geq 0$, of $\mathcal{C}(\boldsymbol{P}(V), E)$ are isomorphic to the Dolbeault cohomology groups $H^{0,q}(\boldsymbol{P}(V), E)$, $q \geq 0$, of E. Since the groups $H^{0,q}(\boldsymbol{P}(V), E)$ vanish for $q \geq 1$ ([4, Theorem 7.3]), we obtain a vanishing theorem for the higher cohomology groups of the complex $\mathcal{C}(\boldsymbol{P}(V), E)$.

The definition of the complex $\mathcal{C}(\boldsymbol{P}(V), E)$ carries over without modifications to submanifolds of finite codimension in $\boldsymbol{P}(V)$ - given a holomorphic vector bundle E over a submanifold X of finite codimension in $\boldsymbol{P}(V)$ and a covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X with Zariski open sets - we define a subcomplex $\mathcal{C}(X, E)$ of the Čech complex corresponding to E and $\mathcal{U}$. We show in Section 6 that if X is a complete intersection (e.g. hypersurface) in $\boldsymbol{P}(V)$ and $\mathcal{U}$ is a suitable covering of X, then the complex $\mathcal{C}(X, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, has an acyclic resolution of finite length. This allows us to prove the vanishing Theorem 6.5: If X is a complete intersection in $\boldsymbol{P}(V)$, $\dim \boldsymbol{P}(V) = \infty$ and $\boldsymbol{P}(V)$ admits smooth partitions of unity, then the higher cohomology groups of the complex $\mathcal{C}(X, E)$ vanish. This vanishing theorem is used in [3] to prove that $H^{0,1}(X, \mathcal{O}_X(n)) = 0$, $n \in \mathbb{Z}$, when X is a complete intersection in an infinite-dimensional complex projective space $\boldsymbol{P}(V)$ which admits smooth partitions of unity.

Let us describe briefly the contents of the paper.

In the book [8] J.-P. Ramis has extended Chow's lemma to all projective spaces modeled on complex Banach spaces. He has proved that if X is a closed analytic set of finite codimension in $\boldsymbol{P}(V)$ for which there exists a fixed number N such that for any $x \in X$ there is a neighbourhood U of x in which $X \cap U$ is the set of common zeros of N holomorphic functions on U, then X is an algebraic set of finite codimension in $\boldsymbol{P}(V)$ [8, Théorème III.2.3.1]. Hence every submanifold of finite codimension n in $\boldsymbol{P}(V)$ is the set of common zeros of a finite number of homogeneous polynomials on V. Since almost all proofs in this paper rest on the algebraic nature of the submanifolds of finite codimension in $\boldsymbol{P}(V)$, Sections 2 and 3 are devoted to the study of infinite-dimensional affine and projective algebraic sets. The results presented in them are well known in finite dimensions but since there was not a suitable reference at hand, it was necessary to give detailed proofs. Our approach is heavily influenced by the book [8] which contains a similar treatment of infinite-dimensional analytic sets.

In Section 4 we consider a finite holomorphic covering $\pi : Y \to Z$ between complex manifolds along with a holomorphic line bundle $L \to Z$ and show that in certain circumstances differential forms on Y with values in $\pi^* L$ can be represented in terms of differential forms on X with values in L. A special case of this representation is used immediately in the proof of Proposition 4.6 which plays important role in Section 6. The general case of Propositions 4.2 and 4.3 is used in [3].

In Section 5 we define the complex $\mathcal{C}(X, E)$ and prove that it is acyclic when $X = \boldsymbol{P}(V)$ admits smooth partitions of unity, and E is a finite rank holomorphic vector bundle over $\boldsymbol{P}(V)$.

In Section 6 we prove the main result of the paper by making use of the Koszul complex in order to construct an acyclic resolution of $\mathcal{C}(X, E)$ when X is a complete intersection in $\boldsymbol{P}(V)$.

This paper is based on the author's Ph.D. thesis (Purdue University, 2001).

2. AFFINE ALGEBRAIC SETS IN BANACH SPACES

Let V be a complex Banach space. A subset $X \subset V$ is an *analytic set of finite codimension* in V, if for any $x \in X$ there exist a neighbourhood U and a finite number of holomorphic functions $\varphi_1, \dots, \varphi_s \in \mathcal{O}(U)$ such that $X \cap U = Z(\varphi_1, \dots, \varphi_s)$. For any open set $U \subset V$ we denote by $\mathcal{I}(X)(U)$ the set of all holomorphic functions on U that vanish on $X \cap U$. The correspondence $U \mapsto \mathcal{I}(X)(U)$ defines a subsheaf $\mathcal{I}(X)$ of $\mathcal{O}_V$. The sheaf $\mathcal{I}(X)$ is an ideal in $\mathcal{O}_V$, which is called the *ideal sheaf* of X. For any $x \in X$ the stalk $\mathcal{I}_x(X)$ of $\mathcal{I}(X)$ at x consists of all holomorphic germs at x that vanish on X in some neighbourhood of x. We say that the point $x \in X$ is *regular*, if there exist a neighbourhood U of x and a finite number of holomorphic functions $\psi_1, \dots, \psi_n \in \mathcal{O}(U)$ such that $X \cap U = Z(\psi_1, \dots, \psi_n)$ and the differentials $\mathrm{d}\psi_1, \dots, \mathrm{d}\psi_n$ are linearly independent at x. By the implicit function theorem the germs $\psi_{1x}, \dots, \psi_{nx}$ generate the ideal $\mathcal{I}_x(X)$, and the tangent space $T_xX = \{\xi \in V : (\mathrm{d}\psi)_x(\xi) = 0 \text{ for all } \psi \in \mathcal{I}_x(X)\}$ to X at x has codimension n in V. The subset X_{reg}, consisting of all regular points of X, is open in X and it is known that X_{reg} is dense in X (see [8]). An analytic set X of finite codimension in V is a *submanifold* of finite codimension in V, if every point of X is regular.

A function $F : V \to \mathbb{C}$ is a homogeneous polynomial of degree d on V if there is a bounded multilinear map $M : V^d \to \mathbb{C}$ such that $F(v) = M(v, \dots, v)$ for any $v \in V$. The vector space of all homogeneous polynomials on V of degree d will be denoted by $\mathbb{C}[V]_d$.

Let X be an analytic set of finite codimension in V, $x_0 \in X$ and $f \in \mathcal{O}_x(V)$, $f \neq 0$. Then there exist a natural number d and unique homogeneous polynomials $F_i \in \mathbb{C}[V]_i$, $i \geq d$, such that $F_d \neq 0$ and $f(x) = \sum_{i \geq d} F_i(x - x_0)$ for all x in some neighbourhood of x_0. The homogenous polynomial F_d is called the *leading term* of the holomorphic germ $f \in \mathcal{O}_x(V)$. The set of common zeros of the leading terms of all holomorphic germs $f \in \mathcal{I}_x(X)$, $f \neq 0$, is called the *tangent cone* C_xX of x at X.

Remark 1. If x is a regular point of X then $C_xX = T_xX$. To see this, we may assume without loss of generality that $x = 0$, $V = V' \times \mathbb{C}^n$ and U and B are neighbourhoods of 0 in V' and $\mathbb{C}^n$ respectively such that

$$X \cap U \times B = \{ (v', Z_1, \dots, Z_n) \in V' \times \mathbb{C}^n : Z_i = \varphi_i(v'),\ i = 1, \dots, n \}$$

where $\varphi_i : U \to \mathbb{C}$, $i = 1, \dots, n$, are holomorphic functions. We may even further assume that all differentials $\mathrm{d}\varphi_i$, $i = 1, \dots n$, vanish at 0, so that $T_xX = V'$. For a given $f \in \mathcal{O}_{V,x}$, let $g \in \mathcal{O}_{V',x}$ be given by $g(v') = f(v', \varphi_1(v'), \dots, \varphi_n(v'))$ for all v' in some neighbourhood of 0 in V'. Let $g = \sum_{i \geq 0} G_{ix}$ with $G_i \in \mathbb{C}[V']_i$. Suppose $f \neq 0$ and let $F_d \in \mathbb{C}[V]_d$ be the leading term of f. Then $G_d = F_d|_{V'}$ because all functions φ_i, $i = 1, \dots n$, vanish of order > 1 at 0. In particular if $f \in \mathcal{I}_x(X)$ and $f \neq 0$ then $F_d|_{V'} = 0$ which yields $C_x X = T_x X$.

A function $f : V \to \mathbb{C}$ is a polynomial on V of degree d if $f = \sum_{k=1}^{d} f_d$, where each f_k, $k = 1, \dots, d$, is a homogeneous polynomial of degree k and $f_d \neq 0$. The ring of all polynomials on V will be denoted by $\mathbb{C}[V]$. Since $\mathbb{C}[V] = \bigoplus_d \mathbb{C}[V]_d$, the ring of all polynomials on V is a graded ring. It is known that $\mathbb{C}[V]$ is a factorial domain (see [8]). In particular the ring $\mathbb{C}[V]$ is integrally closed. For any $f \in \mathbb{C}[V]$ and any $v_0 \in V$ the function $g : V \to \mathbb{C}$ given by $g(v) = f(v + v_0)$, $v \in V$, is also a polynomial on V. Thus we may also speak about polynomials on the *affine* Banach space $\boldsymbol{A}(V)$ associated with V.

A subset X of V is an *algebraic set of finite codimension* in V, if there is a finite number of polynomials $f_1, \dots, f_r \in \mathbb{C}[V]$ such that $X = Z(f_1, \dots, f_r)$. Every algebraic set of finite codimension in V is a closed analytic set of finite codimension in V. The ideal consisting of all polynomials on V that vanish on X is denoted by $I(X)$. The factor-ring $\mathbb{C}[V]/I(X)$ is denoted by $\mathbb{C}[X]$ and is called the *coordinate ring* of X. An algebraic set X of finite codimension in V is said to be *irreducible* if the coordinate ring of X is an integral domain.

Let W be a finite-dimensional complex vector space of dimension n, and let $Z_1, \dots, Z_n$ be a basis of the dual space W^*. It is clear that the ring of all polynomials on $V \times W$ is isomorphic to the polynomial ring $\mathbb{C}[V][Z_1, \dots, Z_n]$. Let X be an algebraic subset of finite codimension in $V \times W$ and let $p : X \to V$ be the restriction of the projection $\pi : V \times W \to V$ to X. We will say that p is a *finite projection* if the homomorphism $p^* : \mathbb{C}[V] \to \mathbb{C}[X]$, given by $\mathbb{C}[V] \hookrightarrow \mathbb{C}[V \times W] \to \mathbb{C}[X]$, is finite, i.e. $\mathbb{C}[X]$ is a finitely-generated $\mathbb{C}[V]$-module. It is easy to see that $p : X \to V$ is a finite projection if and only if for any $1 \leq j \leq n$ there is a monic polynomial $F_j(T) = T^{k_j} + \sum_{i=1}^{k_j} a_i(v) T^{k_j - i} \in \mathbb{C}[V][T]$ such that $F_j(Z_j) \in I(X)$. All fibers $p^{-1}(v)$, $v \in V$, of a finite projection $p : X \to V$ are finite sets. Moreover, for any $v_0 \in V$, there exists a neighbourhood $U \ni v_0$ and a compact set $K \subset W$ such that $p^{-1}(U) \subset U \times K$. This follows from the well known estimate

$$|\alpha| \leq \max\left(1, \sum_{i=1}^{k} |a_i(v)|\right) \tag{2.1}$$

for any root α of a given polynomial $F(v, T) = T^k + \sum_{i=1}^{k} a_i(v) T^{k-i} \in \mathbb{C}[V][T]$. Indeed, for any $v_0 \in V$ there is a neighbourhood O such that $|a_i(v) - a_i(v_0)| < 1$ for any $v \in O$ and any $i = 1, \dots, k$. If $v \in O$ and α is a root of $F(v, T)$ then $|\alpha| < k + \sum_{i=1}^{k} |a_i(v_0)|$ by estimate (2.1). Hence there is a neighbourhood U of v_0 such that all functions $Z_1|_X, \dots, Z_n|_X$ are bounded on $p^{-1}(U)$ which is equivalent to $p^{-1}(U)$ being contained in $U \times K$ for some compact set $K \subset W$. This shows that for any compact set $K' \subset V$ the preimage $p^{-1}(K')$ is compact, i.e. $p : X \to V$ is a *proper* map. In particular p is a closed map: if B is a closed subset of X, then $p(B)$ is a closed subset of V. Hence if $v_0 \in V$ and $N \subset X$ is a neighbourhood of the fiber $p^{-1}(v_0)$, then there is a neighbourhood $U \ni v_0$ such that $p^{-1}(U) \subset N$.

Proposition 2.1. Let X be an algebraic set of finite codimension in $V \times W$ for which $p : X \to V$ is a finite projection. Then the image $p(X)$ of p is an algebraic set of finite codimension in V and $I(p(X)) = I(X) \cap \mathbb{C}[V]$. In particular, a finite projection $p : X \to V$ is surjective if and only if $I(X) \cap \mathbb{C}[V] = (0)$.

Proof. In the proof we assume that the complex space W is one-dimensional because, as soon as the claim is known to be true for such spaces, the general case follows immediately by induction on the complex dimension of W. Let $Z \in W^*$ be a basis of W^*. Since $\mathbb{C}[X]$ is a finite extension of $\mathbb{C}[V]$, there are polynomials $g_1, \dots, g_r \in \mathbb{C}[V][Z]$ such that $X = Z(g_1, \dots, g_r)$ and the leading coefficient of at least one of them is 1. Now we use the existence of a resultant system of several polynomials in a single variable (see [9]):

Let $f_1, \dots, f_r$ be r polynomials of given degrees in a single variable with indeterminate coefficients. Then there exists a system $D_1, \dots, D_h$ of integral polynomials in these coefficients with the property that if these coefficients are assigned values from the field K the conditions $D_1 = 0, \dots, D_h = 0$ are necessary and sufficient in order that either the equations $f_1 = 0, \dots, f_r = 0$ have a solution in a suitable extension field, or that the formal leading coefficients of all polynomials $f_1, \dots, f_r$ vanish.

Let $D_1, \dots, D_h$ be a resultant system of $g_1, \dots, g_r$. Let $d_1, \dots d_h \in \mathbb{C}[V]$ be the system obtained from $D_1, \dots, D_h$ after substituting the coefficients of $g_1, \dots, g_r$ in $D_1, \dots, D_h$. Then the image $p(X)$ coincides with the set of common zeros of the polynomials $d_1, \dots, d_h$. □

We observe that if $\emptyset \neq X \subset V \times W$ is the set of common zeros of r polynomials in $\mathbb{C}[V \times W]$ and the projection $p : X \to V$ is finite then $\dim W \leq r$. Indeed, if $\dim W > r$ the dimension of every non-empty fiber of p would be positive.

Now we are going to prove the normalisation lemma for algebraic subsets of finite codimension in V. Let W be a closed complex vector subspace of V. We will say that a complex vector subspace $V' \subset V$ is complementary to W, or that V' and W are complementary, if V' is closed and the natural linear map $V' \times W \to V$ is an isomorphism.

Proposition 2.2. (The Normalisation Lemma) Let X be a non-empty algebraic set of finite codimension in V, and let (V_0', W_0), $\dim W_1 < \infty$, be a pair of complementary complex vector subspaces of V such that the projection $p_0 : X \to V_0'$ is finite. Then there is a pair (V', W), $\dim W < \infty$, of complementary complex vector subspaces of V such that $W \supset W_0$, $V' \subset V_0'$, and the projection $p : X \to V'$ is finite and surjective.

Proof. Let $X = Z(f_1, \dots f_r)$ with $f_1, \dots, f_r \in \mathbb{C}[V]$. Denote by S be the set of all pairs (V', W) of complementary complex vector subspaces of V such that $\dim W < \infty$, $W \supset W_0$, $V' \subset V_0$, and the projection $p : X \to V'$ is finite. It is clear that $\dim W \leq r$ for any $(V', W) \in S$. Let $(V', W) \in S$ be a pair for which $\dim W$ is maximal. Suppose that $p(X) \neq V'$. Then there is a polynomial $f \in I(X) \cap \mathbb{C}[V']$ with a leading term $f_d \neq 0$. Choose a vector $v' \in V'$ and a bounded linear functional T on V' such that $f_d(v') = 1$ and $T(v') = 1$. Then $V' \cong U' \times \{\mathbb{C}v\}$, where $U' = \operatorname{Ker} T$, and $f = T^d + a_1 T^{d-1} + \cdots + a_d$ with $a_1, \dots, a_d \in \mathbb{C}[U']$. Now the projection $p(X) \to U'$ along $\{\mathbb{C}v'\}$ is finite, which implies that the projection $X \to U'$ along $\{\mathbb{C}v'\} + W$ is also finite. Hence $(U', \{\mathbb{C}v'\} + W) \in S$, which contradicts the assumption that W has maximal dimension. Thus $p(X) = V'$ and the pair (V', W) has the required properties. □

Definition. Let X be an algebraic set of finite codimension in a complex Banach space V. We will say that the pair (V', W) of complex vector subspaces of V is an *admissible factorisation* for X, if $\dim W < \infty$, V' is complementary to W, and the projection map $p : X \to V'$ is finite and surjective.

The Normalisation Lemma shows that admissible factorisations exist for any non-empty algebraic set X of finite codimension in V.

Let X be an irreducible algebraic set of finite codimension in $V \times W$ such that (V, W) is an admissible factorisation for X. Then the homomorphism $p^* : \mathbb{C}[V] \to \mathbb{C}[X]$ is injective and the field of fractions L of $\mathbb{C}[X]$ is a finite extension of the field of fractions K of $\mathbb{C}[V]$. We note that for any $z \in \mathbb{C}[X]$ the coefficients of the minimal polynomial F of z over K belong to $\mathbb{C}[V]$. Indeed, each coefficient of F belongs to K and is integral over $\mathbb{C}[V]$. Since the ring $\mathbb{C}[V]$ is integrally closed, all coefficients of F are in in $\mathbb{C}[V]$. In particular the discriminant D of F also belongs to $\mathbb{C}[V]$. We will use the following well known fact about integral extensions (see [6]): if $z \in \mathbb{C}[V]$ is a generator of L over K then $D\mathbb{C}[X] \subset \mathbb{C}[V][z]$, where D is a discrimininant of the minimal polynomial of z over K.

Proposition 2.3. Let W be a complex vector space of finite dimension n and let X be an irreducible algebraic set of finite codimension in $V \times W$ such that (V, W) is an admissible factorisation for X. Suppose that $Z \in W^*$ is such that $z = Z + I(X) \in \mathbb{C}[X]$ is a generator of the field L over the field K and let $D \in \mathbb{C}[V]$ be the discriminant of the minimal polynomial F of z over the field K. Then $X_D = p^{-1}(V_D)$ is a connected complex submanifold of $V_D \times W$ of codimension n which is dense in X, and $p|_{X_D} : X_D \to V_D$ is a k-sheeted covering map, where $k = [L : K]$.

Proof. Let $e_1, \dots, e_n$ be a basis of W with dual basis $Z_1, \dots, Z_n$ such that

$Z_1 = Z$. Let $z_i = Z_i + I(X) \in \mathbb{C}[X]$, $i = 2, \dots n$. Since $Dz_i \in \mathbb{C}[V][z]$, $i = 2, \dots n$, there are polynomials $F_i(Z) \in \mathbb{C}[V][Z]$, $i = 2, \dots n$ such that $Dz_i = F_i(z)$, $i = 2, \dots, n$. All polynomials $F(Z_1)$ and $DZ_i - F_i(Z)$, $i = 2, \dots, n$, belong to $I(X)$ because $F(z_1) = 0$ and $Dz_i - F_i(z) = 0$, $i = 2, \dots, n$.

We will show first that X_D is the set of all solutions of the equations

$$F(Z_1) = 0\,,\ Z_2 = D^{-1}F_2(Z_1)\,,\ \dots\ ,\ Z_n = D^{-1}F_n(Z_1) \tag{2.2}$$

in $V_D \times W$. Let J (resp. $I(X)_D$) be the ideal generated in $\mathbb{C}[V][Z_1, \dots, Z_n]_D$ by $F(Z_1)$ and $Z_i - D^{-1}F_i(Z_1)$, $i = 2, \dots, n$, (resp. by $I(X)$). It is enough to show that $I(X)_D = J$. We observe that the factor-ring $\mathbb{C}[V][Z_1, \dots, Z_n]_D / J \cong \mathbb{C}[V][z]_D$ is both an integral domain and an integral extension of $\mathbb{C}[V]_D$. Furthermore, the prime ideal $I(X)_D/J$ is such that $I(X)_D/J \cap \mathbb{C}[V]_D = (0)$ (since $I(X) \cap \mathbb{C}[V] = (0)$). This implies $I(X)_D/J = (0)$ because if $A \subset B$ is an integral extension of integral domains and P is a prime ideal in B such that $P \cap A = (0)$, then $P = (0)$ (see [5]). Thus $I(X)_D = J$ and X_D is defined by equations (2.2).

Next we find local solutions of the equation $F(v, Z) = 0$ by means of an integral formula. For a given $v_0 \in V_D$, let α_j, $j = 1, \dots, k$, be the roots of the $F(v_0, Z)$. Choose a positive real number δ such that α_j is the only root of $F(v_0, Z)$ in the disk $|Z - \alpha_j| \leq \delta$, $j = 1, \dots, k$. Let Γ_j be the circle $|Z - \alpha_j| = \delta$, and let $\Gamma \cup_{j=1}^{k} \Gamma_j$. Since F does not vanish on $\{v_0\} \times \Gamma$, there exists a connected neighbourhood $U \subset \boldsymbol{A}(V)_D$ of v_0 such that F does not vanish on $U \times \Gamma$. For $v \in U$ the number of roots of $F(v, Z)$ (counted with multiplicities) lying inside Γ_j is given by the holomorphic function

$$n_j(v) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{F'_Z(v, Z)}{F(v, Z)}\, \mathrm{d}Z\,, \qquad j = 1, \dots, k\,.$$

Since $n_j(v_0) = 1$, the polynomial $F(v, Z)$ has exactly one root $\alpha_j(v)$ lying inside Γ_j for $v \in U$ and this root is given by the holomorphic function

$$\alpha_j(v) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{F'_Z(v, Z)}{F(v, Z)} Z\, \mathrm{d}Z\,, \qquad j = 1, \dots, k\,.$$

Hence $F|_{U \times W} = (Z - \alpha_1(v)) \dots (Z - \alpha_k(v))$. Let $q_j : U \to U \times W$, $j = 1, \dots, k$, be the graph of the holomorphic map $r_j : U \to W$ given by

$$r_j(v) = \alpha_j(v)\, e_1 + D(v)^{-1} \sum_{i=2}^{n} G_i(\alpha_j(v))\, e_i\,, \qquad v \in U\,.$$

Then $p^{-1}(U)$ is the disjoint union of the complex manifolds $q_j(U)$, $j = 1, \dots, k$, and each restriction $p|_{q_j(U)} : q_j(U) \to U$, $j = 1, \dots, k$, is a biholomorphic map.

The set V_D is connected because for any v_1, $v_2 \in V_D$ the complex line joining v_1 and v_2 intersects $Z(D)$ in a finite set. Let C be a connected component of X_D. Then $p|_C : C \to V_D$ is a covering of degree $k_1 \leq k$ and

$$F_1(v, Z) = \prod_{x \in p^{-1}(v) \cap C} (Z - Z(x)) \ \in\ \mathcal{O}(V_D)[Z]\,, \quad v \in V_D\,,$$

is a polynomial of degree k_1 in Z which divides F in the ring $\mathcal{O}(V_D)[Z]$. Since for any $v \in V$ the roots of $F(v, Z)$ are uniformly bounded in some neighbourhood of v, the same is true about the coefficients of F_1, and by the Riemann extension theorem all coefficients of F_1 can be extended to holomorphic functions on V. Moreover, estimate (2.1) shows that the roots of F grow polynomially, i.e. there is a natural number N and a positive real number C such that $|\alpha| \le C(1 + \|v\|)^N$ for any root α of $F(v, Z)$. Hence the coefficients of F_1 also grow polynomially which shows that they are polynomials. We conclude that $F_1 = F$ because F_1 divides the irreducible polynomial F in $\mathbb{C}[V][Z]$. This immediately yields $C = X_D$, and thus X_D is connected.

It remains to prove that X_D is dense in X. Let $x_0 \in X$ and let $O \subset X$ be a neighbourhood of x_0. Choose a real number $\delta > 0$ such that $|Z(x) - Z(x_0)| > \delta$ for all $x \in p^{-1}(p(x_0))$, $x \neq x_0$. Since $O \cup \{x \in X : |Z(x) - Z(x_0)| > \delta\}$ is a neighbourhood of $p^{-1}(p(x_0))$ and the map p is closed, there exists a neighbourhood $U \subset V$ of $p(x_0)$ such that if $p(x) \in U$, then either $x \in O$, or $|Z(x) - Z(x_0)| > \delta$. After shrinking U we may also assume that for any $v \in U$ the polynomial $F(v, Z)$ has a root α such that $|\alpha - Z(x_0)| < \delta$. Let $v \in U \cap V_D$. Then the fiber $p^{-1}(v)$ contains a point x such that $|Z(x) - Z(x_0)| < \delta$, and it is clear that $x \in O$. Thus the set X_D is dense in X. □

Remark 2. Suppose that (V', W) is an admissible factorisation for an irreducible algebraic set X of finite codimension in V. Denote by $i : \mathbb{C}[V] \to \mathbb{C}[X]$ the natural homomorphism given by $i(f) = f + I(X)$, $f \in \mathbb{C}[V]$. Let $Z_1, \ldots, Z_n \in W^*$ be a basis of the dual space W^*. Since the elements $z_1 = i(Z_1), \ldots, z_n = i(Z_n)$ generate the ring $\mathbb{C}[X]$ over the ring $\mathbb{C}[V']$, they also generate the field of fractions L of $\mathbb{C}[X]$ over the field of fractions K of $\mathbb{C}[V']$. Let $A = \{(a_1, \ldots, a_n) \in \mathbb{C}^n : z = a_1 z_1 + \ldots a_n z_n$ is a generator of L over $K\}$. By the theorem for the primitive element, A is a non-empty Zariski open set in $\mathbb{C}^n$. Hence the set $W_g^* = \{Z \in W^* : i(Z)$ generates L over $K\}$ is a non-empty Zariski open set in W^*.

Remark 3. Let (V', W) be an admissible factorisation for an irreducible algebraic set X of finite codimension in V. Proposition 2.3 and Remark 2 show that there exists an open dense subset V_0' of V' such that for any $y \in V_0'$ the fiber $\pi^{-1}(y) = \{y\} \times W$ intersects X transversely in k regular points of X, where $k = [L : K]$.

Corollary 2.4. For any irreducible algebraic set X of finite codimension in V the set X_{reg} is a connected locally closed submanifold of finite codimension in V, which is dense in X. Moreover, for any pair (V', W) of complex vector subspaces of V that is admissible for X, the dimension of W is equal to the codimension of X_{reg} in V.

Proof. Let (V', W) be an admissible factorisation for X. According to Remark 2, there exists $Z \in W^*$ such that $z = Z + I(X) \in \mathbb{C}[X]$ generates the field of fractions L of $\mathbb{C}[X]$ over the field of fractions K of $\mathbb{C}[V']$. Let $F \in \mathbb{C}[V'][Z]$ be the minimal polynomial of Z over K, and let $D \in \mathbb{C}[V']$ be the discriminant of F. Then $X_D \subset X_{reg}$ by Proposition 2.3. Since X_D is connected and dense in X, so is X_{reg}. Proposition 2.3 implies $\operatorname{codim}_V X_{reg} = \operatorname{codim}_V X_D = \dim W$. □

In view of Corollary 2.4 we define the codimension of an irreducible algebraic subset X in V as the codimension of the submanifold X_{reg} in V. The codimension of X in V will be denoted by $\operatorname{codim}_V X$. The next lemma describes the behaviour of the codimension under finite projections.

Lemma 2.5. Let X be an irreducible algebraic set of finite codimension in V, and let (V', W) be a pair of complex vector subspaces of V such that $\dim W < \infty$, V' is complementary to W, and the projection $p : X \to V'$ is finite. Then the set $X' = p(X)$ is an irreducible algebraic set of finite codimension in V', and $\operatorname{codim}_{V'} X' = \operatorname{codim}_V X - \dim W$.

Proof. Let (V'', W') be an admissible factorisation for X' in V'. Then the pair of subspaces $(V', W' \times W)$ is an admissible factorisation for X in V, and Corollary 2.4 yields $\operatorname{codim}_{V'} X' = \dim W' = \dim W' \times W - \dim W = \operatorname{codim}_V X - \dim W$. □

Let X be an irreducible algebraic set of finite codimension n in V. The next two lemmas will be used to prove that for any $x \in X_{reg}$ there exist n polynomials $f_1, \dots, f_n \in I(X)$ such that their differentials $\mathrm{d}f_1, \dots, \mathrm{d}f_n$ are linearly independent at x.

Lemma 2.6. We keep the notation and the assumptions of Proposition 2.3. Suppose that $x_0 = (v_0, w_0) \in X_{reg}$ is such that:
(i) the fiber $\pi^{-1}(\pi(x_0)) = \{v_0\} \times W$ is transversal to X at x_0;
(ii) $Z(x_0) \neq Z(x)$ for all $x \in p^{-1}(p(x_0))$, $x \neq x_0$.
Then $Z(x_0)$ is a simple root of $F(v_0, Z)$.

Proof. It follows from (i) that there exist neighbourhoods $U \subset V$ of v_0, $B \subset W$ of w_0, and a holomorphic map $f : U \to B$ such that $X \cap U \times B = \Gamma(f)$, where $\Gamma(f) \subset U \times B$ is the graph of f. Let δ be a positive real number such that $|Z(x) - Z(x_0)| > \delta$ for all $x \in p^{-1}(v_0)$, $x \neq x_0$, and let $X_\delta = \{x \in X : |Z(x) - Z(x_0)| > \delta\}$. Since p is a proper map, the fiber $p^{-1}(v)$ is contained in $U \times B \cup X_\delta$ for all v that are close to v_0. Let l be the multiplicity of $Z(x_0)$ as a root of $F(v_0, Z)$. For all $v \in V_D$ that are close to v_0, the fiber $p^{-1}(v)$ contains l distinct points x_i, $i = 1, \dots, l$ such that $|Z(x_i) - Z(x_0)| < \delta$ for $i = 1, \dots, l$. Hence for all $v \in V_D$ that are close to v_0, the graph $\Gamma(f)$ contains l distinct points of the fiber $p^{-1}(v)$. This implies $l = 1$. □

Lemma 2.7 Let X be an irreducible algebraic set of finite codimension in

V, and let $V' \subset V$ be a closed complex vector subspace such that $\operatorname{codim}_V V' = \operatorname{codim}_V X$. Then there exists a finite dimensional complex vector subspace $W \subset V$ such that (V', W) is an admissible factorisation for X.

Proof. The claim is true when $\operatorname{codim}_V X = 0$. Suppose that $\operatorname{codim}_V X = n > 0$, and that the lemma is true for all irreducible algebraic subsets of codimension less than n in a Banach space. Let f be a non-zero polynomial in $I(X)$ with leading term f_d. Since the set $V \setminus V'$ is dense in V, there exists a vector $v \in V \setminus V'$ such that $f_d(v) = 1$. Let T be a bounded linear functional on V such that $U = \operatorname{Ker} T \supset V'$ and $T(v) = 1$. Then $\mathbb{C}[V] \cong \mathbb{C}[U][T]$ and $f = T^d + \sum_{i=1}^{d} a_i T^{d-i}$ with $a_i \in \mathbb{C}[U]$, $i = 1, \ldots, d$. Thus the projection $p_1 : X \to U$ along $\{\mathbb{C}v\}$ is finite, and by Lemma 2.4 the set $X' = p_1(X)$ is an irreducible algebraic subset of codimension $n-1$ in U. Since $\operatorname{codim}_U V' = \operatorname{codim}_U X'$, there exists a finite dimensional complex vector subspace $W' \subset U$ such that (V', W') is an admissible factorisation for X' in U. Then $(V', W' \times \{\mathbb{C}v\})$ is an admissible factorisation for X in V, which finishes the proof. □

Proposition 2.8. For any regular point x_0 of an irreducible algebraic set X of finite codimension n in V, there exist polynomials $f_1, \ldots, f_n \in I(X)$ such that their differentials $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ are linearly independent at x_0.

Proof. By Lemma 2.7 with $V' = T_{x_0}X$, there is an n-dimensional subspace W of V such that (V', W) is an admissible factorisation for X in V. Let $Z_1, \ldots, Z_n$ be a basis of W^* that satisfies the following two conditions: (i) $Z_i(x) \neq Z_i(x_0)$ for $x \in p^{-1}(p(x_0))$, $x \neq x_0$, $i = 1, \ldots, n$; (ii) $z_i = Z_i + I(X) \in \mathbb{C}[X]$ generates the field of fractions of $\mathbb{C}[X]$ over the field of fractions of $\mathbb{C}[V']$, $i = 1, \ldots, n$. The existence of such a basis of W^* follows from Remark 2. Let $g_i \in \mathbb{C}[V'][Z_i]$ be the minimal polynomial of z_i over the field of fractions of $\mathbb{C}[V']$, $i = 1, \ldots, n$. Then $\mathrm{d}g_i|_W = g_i'(p(x_0), Z_i(x_0))\,\mathrm{d}Z_i$, $i = 1, \ldots, n$, and since by Lemma 2.6 $g_i'(p(x_0), Z_i(x_0)) \neq 0$, $i = 1, \ldots, n$, the differentials $\mathrm{d}g_1, \ldots, \mathrm{d}g_n$ are linearly independent at x_0. □

3. PROJECTIVE ALGEBRAIC SETS

In this section we first consider a complex Banach space V and briefly describe the complex structure of the corresponding complex projective space $\boldsymbol{P}(V)$. After that we study the properties of algebraic subsets of finite codimension in $\boldsymbol{P}(V)$.

Let V be a complex Banach space with a dual space V^*. The projective space $\boldsymbol{P}(V)$ associated with V consists of all complex lines in V. The set $\boldsymbol{P}(V)$ is given a structure of a complex manifold as follows. For any $v \in V$, $v \neq 0$, the complex line spanned by v will be denoted by $[v]$. For a given bounded linear functional $h \in V^*$, $l \neq 0$, let $\boldsymbol{P}(V)_h = \{[v] \in \boldsymbol{P}(V) : h(v) \neq 0\}$. Denote by A_h the affine hyperplane $A_h = \{v \in V : h(v) = 1\}$, and let $\varphi_h : \boldsymbol{P}(V)_h \to A_h$ be the coordinate map given by $\varphi_h([v]) = h(v)^{-1}v$. The family of sets $\boldsymbol{P}(V)_h$, $h \in V^*$, $h \neq 0$, is a covering of $\boldsymbol{P}(V)$ by the Hahn-Banach theorem, and it is easy to verify that the coordinate maps φ_h, $h \in V^*$, $h \neq 0$, endow $\boldsymbol{P}(V)$ with a structure of a complex manifold.

We note that for any $h \in V^*$, $h \neq 0$, the open set $\boldsymbol{P}(V)_h$ is an affine space. Moreover, for any $F \in \mathbb{C}[V]_d$, the function $f : \boldsymbol{P}(V)_h \to \mathbb{C}$ given by $f([v]) = h(v)^{-d}F(v)$ for $[v] \in \boldsymbol{P}(V)_h$, is a polynomial on $\boldsymbol{P}(V)_h$ because $f \circ \varphi_h^{-1} = F|_{A_h}$ is a polynomial on the affine hyperplane A_h. Since for every polynomial f on A_h there exists a homogeneous polynomial F on V such that $f = F|_{A_h}$, the ring $\mathbb{C}[\boldsymbol{P}(V)_h]$ of all polynomials on $\boldsymbol{P}(V)_h$ is naturally isomorphic to the so called homogeneous localisation $\mathbb{C}[V]_{(h)} = \{F/h^d : F \in \mathbb{C}[V]_d,\ d \geq 0\}$.

A subset $X \subset \boldsymbol{P}(V)$ is an *analytic set of finite codimension* in $\boldsymbol{P}(V)$, if for any $x \in X$ there exist a neighbourhood U and a finite number of holomorphic functions $\varphi_1, \ldots, \varphi_s \in \mathcal{O}(U)$ such that $X \cap U = Z(\varphi_1, \ldots, \varphi_s)$. We say that the point $x \in X$ is *regular*, if there exist a neighbourhood U of x and a finite number of holomorphic functions $\psi_1, \ldots, \psi_n \in \mathcal{O}(U)$ such that $X \cap U = Z(\psi_1, \ldots, \psi_n)$ and the differentials $\mathrm{d}\psi_1, \ldots, \mathrm{d}\psi_n$ are linearly independent at x. The subset X_{reg}, consisting of of all regular points of X, is open in X, and it is known that X_{reg} is dense in X (see [8]). An analytic set X of finite codimension in $\boldsymbol{P}(V)$ is a *submanifold of finite codimension* in $\boldsymbol{P}(V)$, if every point of X is regular.

A subset X of $\boldsymbol{P}(V)$ is an *algebraic set of finite codimension* in $\boldsymbol{P}(V)$, if there is a finite number of homogeneous polynomials $f_1, \ldots, f_r \in \mathbb{C}[V]$ such that X is the set of common zeros of $f_1, \ldots, f_r$ in $\boldsymbol{P}(V)$. Every algebraic set of finite codimension in $\boldsymbol{P}(V)$ is a closed analytic set of finite codimension in $\boldsymbol{P}(V)$. The ideal generated by all homogeneous polynomials on V that vanish on X will be denoted by $I(X)$. Let $I(X)_d$ be the vector space $I(X) \cap \mathbb{C}[V]_d$. Since $I(X) = \bigoplus_{d \geq 0} I(X)_d$, the ideal $I(X)$ is a homogeneous ideal in the graded ring $\mathbb{C}[V]$. The factor-ring $\mathbb{C}[V]/I(X)$ is denoted by $S[X]$ and is called the *homogeneous coordinate ring* of X. Since $I(X)$ is a homogeneous ideal in $\mathbb{C}[V]$, the ring $S[X]$ inherits the grading of $\mathbb{C}[V]$; $S(X) \cong \bigoplus_{d \geq 0} S(X)_d$, where $S(X)_d = \mathbb{C}[V]_d/I(X)_d$. The set $Z(I(X)) \subset V$ is a cone in V and will be denoted by $C(X)$. It is clear that $C(X)$ is an algebraic set of finite codimension in V and $\mathbb{C}[C(X)] = S(X)$.

For any $h \in V^*$, $h \neq 0$, the open set $X_h = X \cap \boldsymbol{P}(V)_h$ is an algebraic set of finite codimension in the affine space $\boldsymbol{P}(V)_h$. If $F \in I(X)_d$, then $F/h^d \in I(X_h)$ and, conversely, if $F/h^d \in I(X_h)$ for some $F \in \mathbb{C}[V]_d$, then $Fh \in I(X)_{d+1}$. Hence the coordinate ring $\mathbb{C}[X_h]$ of X_h is isomorphic to the so called homogeneous localisation $S(X)_{(h)}$.

We will say that X is an *irreducible* algebraic set of finite codimension in $\boldsymbol{P}(V)$ if $S[X]$ is an integral domain. If X is an irreducible algebraic set of finite codimension in $\boldsymbol{P}(V)$, then X_h is an irreducible algebraic set of finite codimension in $\boldsymbol{P}(V)_h$ for any $h \in V^*$, $h \notin I(X)$. It is clear that, for every irreducible algebraic set X of finite codimension in $\boldsymbol{P}(V)$, the family of sets $\mathcal{C} = \{\boldsymbol{P}(V)_h\}$, $h \in V^*$, $h \notin I(X)$, is an open covering of X. Moreover, for any $\boldsymbol{P}(V)_{h_1}, \boldsymbol{P}(V)_{h_2} \in \mathcal{C}$ the intersection $\boldsymbol{P}(V)_{h_1} \cap \boldsymbol{P}(V)_{h_2}$ is dense in both $\boldsymbol{P}(V)_{h_1}$ and $\boldsymbol{P}(V)_{h_2}$. Hence all sets $\boldsymbol{P}(V)_h \in \mathcal{C}$ are dense in X.

Proposition 3.1. For any irreducible algebraic subset X of finite codimension in $\boldsymbol{P}(V)$, the set X_{reg} is a connected open subset of X which is dense in X.

Proof. This follows from Corollary 2.4 and the considerations above. □

In view of Proposition 3.1, we define the codimension of an irreducible algebraic subset X in $\boldsymbol{P}(V)$ as the codimension of the locally closed submanifold X_{reg} in $\boldsymbol{P}(V)$. The codimension of X in $\boldsymbol{P}(V)$ will be denoted by $\operatorname{codim}_{\boldsymbol{P}(V)} X$.

Let W be a finite-dimensional complex vector space of dimension n. The projection map $V \times W \to V$ induces a holomorphic map

$$\pi : \boldsymbol{P}(V \times W) \setminus \boldsymbol{P}(W) \to \boldsymbol{P}(V)$$

given by $p([(v,w)]) = [v]$ for $[(v,w)] \in \boldsymbol{P}(V \times W) \setminus \boldsymbol{P}(W)$. For $h \in V^*$, $h \neq 0$, let $\theta_h : \pi^{-1}(\boldsymbol{P}(V)_h) \to \boldsymbol{P}(V)_h \times W$ be the trivialisation given by $\pi_h([v,w]) = ([v], h(v)^{-1}w)$. It is easy to verify that the family of trivialisations θ_h, $h \in V^*$, $h \neq 0$, makes π into a locally trivial vector bundle over $\boldsymbol{P}(V)$ with fiber W.

Let $Z_1, \ldots, Z_n$ be a basis of W^*. For any algebraic set X of finite codimension in $\boldsymbol{P}(V \times W)$, we denote by $p : X \setminus \boldsymbol{P}(W) \to \boldsymbol{P}(V)$ the restriction of π to $X \setminus \boldsymbol{P}(W)$, and by $p^* : \mathbb{C}[V] \to S[X]$ the ring homomorphism given by the composition $\mathbb{C}[V] \hookrightarrow \mathbb{C}[V][Z_1, \ldots, Z_n] \cong \mathbb{C}[V \times W] \to S[X]$. It is clear that the homomorphism p^* respects the grading of $\mathbb{C}[V]$ and $S[X]$, i.e. $p^*(\mathbb{C}[V]_d) \subset S[X]_d$, $d \geq 0$. Furthermore, for any $h \in V^*$, $h \notin I(X)$, the homomorphism $p_h^* : \mathbb{C}[V]_{(h)} \to \mathbb{C}[X_h]$, associated with the projection $p_h = p|_{X_h} : X_h \to \boldsymbol{P}(V)_h$, is induced by the homomorphism $p^* : \mathbb{C}[V] \to S[X]$ (via homogeneous localisation with respect to h). Hence if p^* is finite, then p_h^* is finite for any $h \in V^*$, $h \notin I(X)$.

Remark 1. If p^* is a finite homomorphism, then all fibers of the map $p_C : C(X) \to V$ are finite sets. In particular the cone $p_C^{-1}(0) = C(X) \cap W$ is a finite set. Hence $C(X) = \cap W\{0\}$, which implies $X \cap \boldsymbol{P}(W) = \emptyset$. The next proposition shows that the converse is also true.

Lemma 3.2. Let X be an algebraic set of finite codimension in $\boldsymbol{P}(V \times W)$ such that $X \cap \boldsymbol{P}(W) = \emptyset$. Then the homomorphism $p^* : \mathbb{C}[V] \to S[X]$ is finite, the set $Y = p(X)$ is an algebraic set of finite codimension in $\boldsymbol{P}(V)$, and $I(Y) = I(X) \cap \mathbb{C}[V]$. If the algebraic set X is irreducible, then the algebraic set Y is also irreducible, and $\operatorname{codim}_{\boldsymbol{P}(V')} Y = \operatorname{codim}_{\boldsymbol{P}(V)} X - \dim W$.

Proof. As in the proof of Proposition 2.1, we may assume that $\dim W = 1$, i.e. $W = \{\mathbb{C}e\}$, $e \neq 0$. Let $Z \in W^*$ be a basis of W^*. Since $[(e,0)] \notin X$, the ideal $I(X)$ contains a homogeneous polynomial $g = Z^n + a_1 Z^{n-1} + \ldots + a_n$, $a_1, \ldots, a_n \in \mathbb{C}[V]$, with leading coefficient 1 with respect to Z. Thus $p^* : \mathbb{C}[V] \to S[X]$ is a finite homomorphism. Let $C(X) \subset V \times W$ (resp. $C(Y) \subset V$) be the cone of X (resp. Y). By Proposition 2.1 the set $C(Y)$ is an algebraic set of finite codimension in V and $I(C(Y)) = I(C(X)) \cap \mathbb{C}[V]$. Hence Y is an algebraic set of finite codimension in $\boldsymbol{P}(V)$ and $I(Y) = I(X) \cap \mathbb{C}[V']$. If X is irreducible, then $I(X) \cap \mathbb{C}[V']$ is a prime ideal in $\mathbb{C}[V']$, and Y is also irreducible. Let h be a bounded linear functional on V' such that $Y_h = \boldsymbol{P}(V')_h \cap Y \neq \emptyset$. Since the projection $p|_{X_h} : X_h \to \boldsymbol{P}(V')_h$ is finite, $\operatorname{codim}_{\boldsymbol{P}(V')_h} Y_h = \operatorname{codim}_{\boldsymbol{P}(V)_h} X_h - \dim W$ by Lemma 2.5. This proves the last claim of this lemma because X_h and Y_h are open and dense in X and Y respectively. □

The Normalisation Lemma has a natural analogue for projective spaces:

Proposition 3.3. (The Projective Normalisation Lemma) Let X be an algebraic set of finite codimension in $\boldsymbol{P}(V)$ and W_0 be a finite dimensional complex vector subspace such that $\boldsymbol{P}(W_0) \cap X = \emptyset$. Then there is a finite dimensional complex vector subspace $W \supset W_0$ such that for any complementary complex vector subspace V' the projection map $p : X \to \boldsymbol{P}(V')$ is surjective.

Proof. Let V_0' be a complex vector subspace which is complementary to W_0. Since the homomorphism $p_0^* : \mathbb{C}[V_0'] \to S[X] = \mathbb{C}[C(X)]$ is finite by Lemma 3.2, we can apply Proposition 2.2 to the pair (V_0', W_0) and the cone $C(X)$. Thus there exists a pair of complementary complex vector subspaces (V', W), $\dim W < \infty$, $W \supset W_0$, $V' \subset V_0'$, for which the homomorphism $p^* : \mathbb{C}[V'] \to S(X)$ is finite and $I(X) \cap \mathbb{C}[V'] = (0)$. Remark 1 shows that $\boldsymbol{P}(W) \cap X = \emptyset$, and Lemma 3.2 shows that the projection map $p : X \to \boldsymbol{P}(V')$ is surjective. If V_1' is another complex subspace that is complementary to W, then

$$\mathbb{C}[V_1'] = \{f \in \mathbb{C}[V] : f(v + w) = f(v) \text{ for any } w \in W\} = \mathbb{C}[V'],$$

which shows that the projection map $p_1 : X \to \boldsymbol{P}(V_1')$ is surjective too. □

Definition. Let X be an algebraic set of finite codimension in a projective space $\boldsymbol{P}(V)$. Given a pair (V', W) of complementary complex vector subspaces of V, we say that (V', W) is an *admissible factorisation* for X if $\dim W < \infty$, $\boldsymbol{P}(W) \cap X = \emptyset$, and the projection $p : X \to \boldsymbol{P}(V')$ is surjective.

The Projective Normalisation Lemma shows that admissible factorisations exist for any algebraic set X of finite codimension in $\boldsymbol{P}(V)$. If X is irreducible, then a given pair (V', W) of complementary complex vector subspaces is an admissible factorisation for X if and only if $\boldsymbol{P}(W) \cap X = \emptyset$ and $\dim W = \operatorname{codim}_{\boldsymbol{P}(V)} X$.

We note that if the pair (V, W) is an admissible factorisation for an irreducible algebraic set X of finite codimension in $\boldsymbol{P}(V \times W)$, then $p^* : \mathbb{C}[V] \to S[X]$ is an injective homomorphism, and the field of fractions L of $S(X)$ is a finite extension of the field of fractions K of $\mathbb{C}[V]$. Now we are going to prove an analogue of Proposition 2.3 for irreducible algebraic sets of finite codimension in projective space.

Proposition 3.4. Let W be a complex vector space of finite dimension n and let X be an irreducible algebraic set of finite codimension in $\boldsymbol{P}(V \times W)$ such (V, W) is an admissible factorisation for X. Suppose that $Z \in W^*$ is such that $z = Z + I(X) \in S[X]$ is a generator of the field L over the field K, and let $D \in \mathbb{C}[V]$ be the discriminant of the minimal polynomial of z over K. Then $X_D = p^{-1}(\boldsymbol{P}(V)_D)$ is a complex submanifold of $\boldsymbol{P}(V \times W)_D$ of codimension n and $p|_{X_D} : X_D \to \boldsymbol{P}(V)_D$ is a d-sheeted covering map, where $d = [L : K]$.

Proof. We note that for a given $h \in V^*$, $h \neq 0$, the fraction z/h generates the field of fractions L_h of $S[X]_{(h)}$ over the field of fractions K_h of $\mathbb{C}[V]_{(h)}$. Let D_h be the discriminant of the minimal polynomial of z/h over K_h. According to Proposition 2.3, the set $(X_h)_{D_h}$ is a complex submanifold of $(\boldsymbol{P}(V)_h)_{D_h} \times W$ of codimension n and the map $p_h|_{(X_h)_{D_h}} : (X_h)_{D_h} \to (\boldsymbol{P}(V)_h)_{D_h}$ is a k_h-sheeted covering map, where $k_h = [L_h : K_h]$. Since $K = K_h(h)$, $L = L_h(h)$, and h is transcendental over K_h, we see that $k_h = d$ for $h \in V^*$, $h \neq 0$. A simple calculation shows that $D_h = D/h^{k(k-1)}$ which implies $(\boldsymbol{P}(V)_h)_{D_h}(\boldsymbol{P}(V)_D)_h$. To finish the proof, we observe that the family of sets $(\boldsymbol{P}(V)_D)_h$, $h \in V^*$, $h \neq 0$, is an open covering of $\boldsymbol{P}(V)_D$. □

Definition. Let (V', W) be an admissible factorisation for an irreducible algebraic set X of finite codimension in $\boldsymbol{P}(V)$ and $z \in S(X)_1$. We will say that (W, V', z) is an *admissible triple* for X, if z is a generator of the field of fractions L of $S(X)$ over the field of fractions K of $\mathbb{C}[V]$. Given an admissible triple (W, V', z) for X, the discriminant of the minimal polynomial of z over K is denoted by D.

Remark 2. According to Remark 2 in Section 2, if (V', W) is an admissible factorisation for an irreducible algebraic set X of finite codimension in $\boldsymbol{P}(V)$, then there is an element $z \in S(X)_1$ such that (V', W, z) is an admissible triple for X.

If (W, V', z) is an admissible triple for an irreducible algebraic set X of finite codimension in $\boldsymbol{P}(V)$ and $y \in \boldsymbol{P}(V')_D$, then the fiber $\pi^{-1}(y) = \{y\} \times W$ intersects the manifold X_{reg} transversely in d points, where $d = [L : K]$. Let $U \subset V$ be the complex vector subspace spanned by W and y in V. Then $\dim \boldsymbol{P}(U) = \operatorname{codim}_{\boldsymbol{P}(V)} X$ and $\boldsymbol{P}(U)$ intersects the manifold X_{reg} transversely in d points. The next lemma shows that the number d is the same for all admissible triples for X in $\boldsymbol{P}(V)$.

Lemma 3.5. Let X be an irreducible algebraic set of finite codimension n in $\boldsymbol{P}(V)$, and let U_1, U_2 be $n+1$-dimensional complex vector subspaces of V such that both sets $\boldsymbol{P}(U_1) \cap X$, $\boldsymbol{P}(U_2) \cap X$ consist of regular points of X and both intersections $\boldsymbol{P}(U_1) \cap X_{reg}$, $\boldsymbol{P}(U_2) \cap X_{reg}$ are transversal. Then the cardinality of the finite sets $\boldsymbol{P}(U_1) \cap X$ and $\boldsymbol{P}(U_2) \cap X$ is the same.

Proof. If $\dim V < \infty$, then the lemma is true because the cardinality of both sets $\boldsymbol{P}(U_1) \cap X$ and $\boldsymbol{P}(W_2) \cap X$ is equal to the degree of X in $\boldsymbol{P}(V)$. Let V' be a finite dimensional complex vector subspace of V which contains both U_1 and U_2 and let $X' = X \cap \boldsymbol{P}(V')$. Then both sets $\boldsymbol{P}(U_1) \cap X'$, $\boldsymbol{P}(W_2) \cap X'$ consist of regular points of X' and both intersections $\boldsymbol{P}(U_1) \cap X'_{reg}$, $\boldsymbol{P}(U_2) \cap X'_{reg}$ are transversal. Hence the cardinality of the sets $\boldsymbol{P}(U_1) \cap X$, $\boldsymbol{P}(U_2) \cap X$ is the same. □

In view of Lemma 3.5 the following definition makes sense.

Definition. Let X be an irreducible algebraic set of finite codimension in $\boldsymbol{P}(V)$. The *degree* of X in $\boldsymbol{P}(V)$ is the degree of the field of fractions of $S(X)$ over the field of fractions of $\mathbb{C}[V']$ for any admissible factorisation (V', W) for X. The degree of X in $\boldsymbol{P}(V)$ will be denoted by $\deg X$.

Remark 3. Let X be an irreducible algebraic set X of finite codimension $n < \infty$ and degree d in $\boldsymbol{P}(V)$. Then for any given point $x_0 \in \boldsymbol{P}(V) \setminus X$ there exists an $n+1$-dimensional complex vector subspace U of V such that $\boldsymbol{P}(U)$ passes through x_0, and $\boldsymbol{P}(U)$ intersects X transversely in d distinct regular points of X. Indeed, let (V', W, z) be an admissible triple for X such that $x_0 \in \boldsymbol{P}(W)$ (see Proposition 3.3 and Remark 2). Then for any $y \in \boldsymbol{P}(V')_D$ the complex vector subspace U, spanned by W and y in V, has the required properties.

Now we are going to show that if x_0 is a regular point of an irreducible algebraic subset X of finite codimension n and degree d in $\boldsymbol{P}(V)$, then there exists an $n+1$-dimensional complex vector subspace U of V such that $\boldsymbol{P}(U)$ passes through x_0, and $\boldsymbol{P}(U)$ intersects X transversely in d distinct regular points of X.

Let x_0 be a regular point of an irreducible algebraic set X of finite codimension in $\boldsymbol{P}(V)$ and let $l \subset \boldsymbol{P}(V)$ be a projective line passing through x_0. We say that l is a *tangent line* to X at x_0 if $f|_l$ vanishes of order > 1 at x_0 for any $f \in I(X)$. The union of all tangent lines to X at x_0 will be denoted by $P_{x_0}X$. It is clear that the set $P_{x_0}X$ is a projective subspace of codimension n in $\boldsymbol{P}(V)$. Let V' be a closed hyperplane in V such that $x_0 \notin \boldsymbol{P}(V')$ and let $\pi : \boldsymbol{P}(V) \setminus \{x_0\} \to \boldsymbol{P}(V')$ be the map induced by the projection $p_1 : V \cong V' \times x_0 \to V'$. Let $Z : V \to \mathbb{C}$ be a bounded linear functional on V such that $\operatorname{Ker} Z = V'$. Then for any given homogeneous polynomial $0 \neq f \in \mathbb{C}[V]$ there are unique homogeneous polynomials $a_i(f) \in \mathbb{C}[V']$, $i = 0, \ldots, m$, such that $f = \sum_{i=0}^{m} a_i(f)\, Z^{m-i}$, $a_0 \neq 0$, and $\deg a_i + m - i = \deg f$, $i = 0, \ldots, m$.

Lemma 3.6. Let x_0 be a regular point of an irreducible algebraic set X of finite codimension n in $\boldsymbol{P}(V)$ and V' be a hyperplane in V such that $x_0 \notin \boldsymbol{P}(V')$.

Then

$$\pi(P_{x_0}X \setminus \{x_0\}) = \{ [v'] \in \boldsymbol{P}(V') : a_0(f)(v') = 0 \text{ for all } f \in \cup_{d\geq 0} I(X)_d \setminus \{0\} \}.$$

Proof. Let $v_0 \in x_0$ be such that $Z(x_0) = 1$. Since $\boldsymbol{P}(V)_Z$ is by definition the affine subspace $v_0 + V' \subset V$, we will identify the tangent space $T_{x_0}\boldsymbol{P}(V)$ with V'. Then $P_{x_0}X \setminus \boldsymbol{P}(V') = \{[v', v_0] \in \boldsymbol{P}(V)_Z : v' \in T_{x_0}X\}$ and $\pi(P_{x_0}X \setminus \{x_0\}) = \boldsymbol{P}(T_{x_0}X)$. Thus it is enough to prove that

$$T_{x_0}X = \{ v' \in V' : a_0(f)(v') = 0 \text{ for all } f \in \cup_{d\geq 0} I(X)_d \setminus \{0\} \}. \tag{3.1}$$

Denote by $A_{x_0}X$ the set on the right-hand side of (3.1). For $f \in \cup_{d\geq 0} I(X)_d \setminus \{0\}$, let $g \in I(X_Z)$ be given by $g([v', v_0]) = f(v', v_0) = \sum_{i=0}^{m} a_i(f)(v')$, $v' \in V'$. Then $g(x) = \sum_{i\geq 0} a_i(f)(x - x_0)$ for all $x \in \boldsymbol{P}(V)_Z$, which shows that $a_0(f)$ is the leading term of the holomorphic germ $g_{x_0} \in \mathcal{I}_{x_0}(X_Z)$. Hence $A_{x_0}X \supset C_{x_0}X$, where $C_{x_0}X$ is the tangent cone of X at x_0. According to Proposition 2.8, there exist polynomials $g_j \in I(X_Z)$, $j = 1, \ldots, n$ such that the differentials $\mathrm{d}g_1, \ldots, \mathrm{d}g_n$ are linearly independent at x_0. Let $g_j([v', v_0]) = \sum_{i=0}^{m_j} a_{ij}(v')$, $v' \in V'$, where $a_{ij} \in \mathbb{C}[V']_{i+1}$ for $j = 1, \ldots, n$, $0 \leq i \leq m_j$. Then the linear functionals $a_{01}, \ldots, a_{0n}$ are exactly the differentials of $g_1, \ldots, g_n$ at x_0. Let $f_j \in \mathbb{C}[V]_{m_j+1}$ be given by $f_j(v) = \sum_{i=0}^{m_j} a_{ij}(p_1(v))\, Z(v)^{m_j-i}$, $v \in V$, $j = 1, \ldots, n$. Then $f_j \in I(X)_{m_j+1}$ and $a_0(f_j) = a_{0j}$, $j = 1, \ldots, n$. Hence $T_{x_0}X \supset A_{x_0}X$. Since $T_{x_0}X = C_{x_0}X$ (see Remark 1 in Section 2), we conclude that $A_{x_0}X = T_{x_0}X$. □

In the next lemma we keep the notation and the assumptions of Lemma 3.6.

Lemma 3.7. Let $Y = \pi(X \setminus \{x_0\}) \cup \pi(P_{x_0} \setminus \{x_0\})$. Then Y is an irreducible algebraic set of finite codimension in $\boldsymbol{P}(V')$ such that $I(Y) = I(X) \cap \mathbb{C}[V']$. If $h : V' \to \mathbb{C}$, $h \notin I(Y)$, is a bounded linear functional which vanishes on $\pi(P_{x_0} \setminus \{x_0\})$, then $\pi|_{X_h} : X_h \to \boldsymbol{P}(V')_h$ is a finite map (in the sense that the homomorphism $p_{(h)} : \mathbb{C}[V]_{(h)} \to S(X)_{(h)}$ is finite).

Proof. Let $f_j \in I(X)$, $j = 1, \ldots, n$, be the polynomials which were defined in the proof of Lemma 3.6. Choose homogeneous polynomials $G_j = \sum_{i=0}^{m_j} b_{ij}\, Z^{m_j-i} \in I(X)$, $j = 1, \ldots, r$ such that $G_j = f_j$ for $j = 1, \ldots, n$, and $X = Z(F_1, \ldots, F_r)$. Lemma 3.6 shows that $Z(b_{01}, \ldots, b_{0r}) = \pi(P_{x_0} \setminus \{x_0\})$. Let $D_1, \ldots, D_h \in \mathbb{C}[V']$ be a resultant system of $G_1, \ldots, G_r$ (see the proof of Proposition 2.1). The properties of $D_1, \ldots, D_h$ imply $Z(D_1, \ldots, D_h) = \pi(X \setminus \{x_0\}) \cup Z(b_{01}, \ldots, b_{0r}) = Y$, which proves that Y is an algebraic set of finite codimension in $\boldsymbol{P}(V')$.

We note that $a_0(f) = f$ for any homogeneous polynomial $f \in I(X) \cap \mathbb{C}[V']$. Hence any $f \in I(X) \cap \mathbb{C}[V']$ vanishes on both sets $\pi(X \setminus \{x_0\})$ and $\pi(P_{x_0} \setminus \{x_0\})$, which yields $I(X) \cap \mathbb{C}[V'] \subset I(Y)$. Since any $f \in I(Y)$ vanishes on X, we obtain $I(Y) = I(X) \cap \mathbb{C}[V']$.

Suppose that $h \in V'^*$ vanishes on $\pi(P_{x_0} \setminus \{x_0\}) = Z(b_{01}, \ldots, b_{0n})$. Then h belongs to the subspace which is spanned by the linear functionals $h_j = b_{0j}$, $j = 0, \ldots, n$, in V'^*, which shows that the ideal generated by h_j/h, $j = 1, \ldots, n$, in $\mathbb{C}[V]_{(h)}$ is exactly $\mathbb{C}[V]_{(h)}$. Thus, in order to prove that the homomorphism $p_{(h)}$, is finite, it is enough to prove that all localisations $(p_{(h)})_{hj/h} : (\mathbb{C}[V']_{(h)})_{hj/h} \to (S(X)_{(h)})_{hj/h}$, $j = 1, \ldots, n$, of $p_{(h)}$ are finite. In view of the natural isomorphisms $(\mathbb{C}[V']_{(h)})_{hj/h} \cong (\mathbb{C}[V']_{(h_j)})_{h/h_j}$ and $(S(X)_{(h)})_{hj/h} \cong (S(X)_{(h_j)})_{h/h_j}$, we see that it is enough to prove that all homomorphisms $p_{(h_j)} : \mathbb{C}[V']_{(h_j)} \to S(X)_{(h_j)}$, $j = 1, \ldots, n$, are finite. To this end we note first, that $\mathbb{C}[V]_{(h_j)} \cong \mathbb{C}[V']_{(h_j)}[Z/h_j]$, $j = 1, \ldots, n$. Let $d_j = \deg f_j$ $j = 1, \ldots, n$. Then $h_j^{-d_j} f_j \in \mathbb{C}[V']_{(h_j)}[Z/h_j]$ is a monic polynomial which belongs to $I(X_{h_j})$, $j = 1, \ldots, n$, whence all homomorphisms $p_{(h_j)} : \mathbb{C}[V']_{(h_j)} \to S(X)_{(h_j)}$, $j = 1, \ldots, n$, are finite. □

Lemma 3.8. For any regular point x_0 of an irreducible algebraic subset X of finite codimension n and degree d in $\boldsymbol{P}(V)$, there exists an $n+1$-dimensional complex subspace U of V such that $\boldsymbol{P}(U)$ passes through x_0 and $\boldsymbol{P}(U)$ intersects X transversely in d regular points of X.

Proof. Since the claim is obvious when X is a linear projective subspace of finite codimension in $\boldsymbol{P}(V)$, we will assume that $\deg X > 1$. Let V' be a closed hyperplane in $\boldsymbol{P}(V)$ such that $x_0 \notin \boldsymbol{P}(V')$ and let $\pi : \boldsymbol{P}(V) \setminus \{x_0\} \to \boldsymbol{P}(V')$ be the map induced by the projection $p_1 : V \cong V' \times x_0 \to V'$. Then according to Lemma 3.7, the set $Y = \pi(X \setminus \{x_0\}) \cup \pi(P_{x_0} \setminus \{x_0\})$ is an irreducible algebraic set of codimension $n-1$ in $\boldsymbol{P}(V')$, and the set $\pi(P_{x_0}X \setminus \{x_0\})$ is a linear projective subspace of codimension n in $\boldsymbol{P}(V)$. Let (V'', W') be an admissible factorisation for Y in $\boldsymbol{P}(V')$ and let W be the n-dimensional vector subspace of V which is spanned by W'' and x_0. Then $\boldsymbol{P}(W) \cap (X \cup P_{x_0}X) = \{x_0\}$ because $\boldsymbol{P}(W') = \cap Y\emptyset$. Denote by $\pi' : \boldsymbol{P}(V) \setminus \boldsymbol{P}(W) \to \boldsymbol{P}(V'')$ (resp. $\pi'' : \boldsymbol{P}(V') \setminus \boldsymbol{P}(W') \to \boldsymbol{P}(V'')$) the map induced by the projection$V = V'' \times W \to V''$ (resp. $V' = V'' \times W' \to V''$). Since the set $\pi'(P_{x_0}X \setminus \{x_0\})$ is a projective hyperplane in $\boldsymbol{P}(V'')$, there exists a linear functional $h' \in V''^*$ such that $Z(h') = \pi'(P_{x_0}X \setminus \{x_0\})$. Let $h \in V'^*$ be given by the composition $V' = V'' \times W' \to V'' \xrightarrow{h'} \mathbb{C}$. Then h vanishes on $\pi(P_{x_0} \setminus \{x_0\})$ and the map $\pi|_{X_h} : X_h \to \boldsymbol{P}(V')_h$ is finite by Lemma 3.7. Taking into account that the map $\pi''|_Y : Y \to \boldsymbol{P}(V'')$ is finite and surjective, we conclude that $\pi'|_{X_h} : X_h \to \boldsymbol{P}(V'')_{h'}$ is a finite surjective map. Hence $\boldsymbol{P}(V)_h \cong \boldsymbol{P}(V'')_{h'} \times W$ is an admissible factorisation for X_h in $\boldsymbol{P}(V)_h$. According to Remark 3 in Section 2, there is a point $y \in \boldsymbol{P}(V'')_h$ such that the set ${\pi'}^{-1}(y) \cap X_h$ consists of regular points of X_h, and the intersection of ${\pi'}^{-1}(y)$ and X_h is transversal at each of these points. We also note that ${\pi'}^{-1}(y) \cap X_h {\pi'}^{-1}(y) \cap X$ and ${\pi'}^{-1}(y) \cap P_{x_0}X = \emptyset$ because ${\pi'}^{-1}(y) \subset \boldsymbol{P}(V)_h$. Let U be the $n+1$-dimensional complex vector subspace spanned by W and y in V. Since $\boldsymbol{P}(U) = {\pi'}^{-1}(y) \cup \boldsymbol{P}(W)$, we obtain $\boldsymbol{P}(U) \cap X = ({\pi'}^{-1}(y) \cap X_h) \cup \{x_0\}$ and $\boldsymbol{P}(U) \cap P_{x_0}X = \{x_0\}$. Hence the set $\boldsymbol{P}(U) \cap X$ consists of regular points of X

and the intersection of $\boldsymbol{P}(U)$ and X is transversal at each of those points. $\square$

Proposition 3.9. If X is a submanifold of finite codimension in $\boldsymbol{P}(V)$, then for any $x_0 \in \boldsymbol{P}(V)$ there exists an admissible triple (V', W, z) for X such that $x_0 \notin \boldsymbol{P}(W)$ and $\pi(x_0) \in \boldsymbol{P}(V')_D$.

Proof. Let $n = \operatorname{codim}_{\boldsymbol{P}(V)} X$ and $d = \deg X$. Choose an $n+1$-dimensional complex vector subspace U of V which passes through x_0 and intersects X transversely in d regular points x_i, $i = 1, \dots, d$, of X (see Remark 3 and Lemma 3.8). Choose an n-dimensional complex vector subspace W of U such that $x_i \notin \boldsymbol{P}(W)$, $i = 0, \dots, d$. Let V' be a vector subspace of V which is complementary to W. Then (V', W) is an admissible factorisation for X and $\pi^{-1}(\pi(x_0)) = \boldsymbol{P}(U) \setminus \boldsymbol{P}(W)$. Let $y = \pi(x_0) = [v_0']$, $v_0' \in V'$, and let $x_i = [v_0' + w_i]$, $w_i \in W$, $i = 1, \dots, d$. Choose a linear functional $Z : W \to \mathbb{C}$ such that $Z(w_i) \neq Z(w_j)$ for $i \neq j$, and set $z = Z + I(X) \in S(X)$. Let $F(v', Z) \in \mathbb{C}[V'][Z]$ be the minimal polynomial of z over the field of fractions of $\mathbb{C}[V']$. Since $F(v_0', Z(w_i)) = 0$, $i = 1, \dots, d$, the degree of the polynomial F is equal to d. Hence z is a generator of the field of fractions of $S(X)$ over the field of fractions of $\mathbb{C}[V]$ and $\pi(x_0) \in \boldsymbol{P}(V')_D$. $\square$

Corollary 3.10. For any submanifold X of finite codimension in $\boldsymbol{P}(V)$, there exists a family of admissible triples $\{(V_i', W_i, z_i)\}_{i \in I}$, for which the family of open sets $\{\boldsymbol{P}(V)_{D_i}\}_{i \in I}$ is an open covering of $\boldsymbol{P}(V)$.

Proof. This is just a rephrasing of the previous proposition. $\square$

4. A REPRESENTATION THEOREM FOR DIFFERENTIAL FORMS

In this section, we consider a submanifold X of finite codimension in $\boldsymbol{P} = \boldsymbol{P}(V)$ and an admissible triple (V', W, z) for X in $\boldsymbol{P}$. We note that $p^* \mathcal{O}_{\boldsymbol{P}(V')}(k) = \mathcal{O}_X(k)$, where $\mathcal{O}_X(k)$ is the restriction of the line bundle $\mathcal{O}_{\boldsymbol{P}}(k)$ to X, $k \in \mathbb{Z}$. According to Proposition 3.4, the map $p|_{X_D} : X_D \to \boldsymbol{P}(V')_D$ is a finite covering of degree $d = \deg X$. We will show that, for any differential form $g \in C^r_{p,q}(X_D, \mathcal{O}_X(k))$, there exist unique differential forms $g_j \in C^r_{p,q}(\boldsymbol{P}(V')_D, \mathcal{O}_{\boldsymbol{P}(V')}(k-j))$, $j = 0, \dots, d-1$, such that

$$g = \sum_{j=0}^{d-1} (p|_{X_D})^* g_j \otimes z^j . \tag{4.1}$$

Representation (4.1) will be derived in a more general setting. Let Y and Z be complex manifolds and let $\pi : Y \to Z$ be a covering map of finite degree d. Let $L \to Z$ be a given holomorphic line bundle over Z and let $M \to Y$ be the line bundle $\pi^* L$. The ring $\oplus_{n \in \mathbb{Z}} H^0(Z, L^n)$ will be denoted by S.

Proposition 4.1. Let $s \in H^0(Y, M)$ and $a_i \in H^0(Z, L^i)$, $i = 1, \dots, d$, be such that $s^d + (\pi^* a_1) s^{d-1} + \dots + (\pi^* a_{d-1}) s + \pi^* a_d = 0$. If the discriminant $D \in$

$H^0(Z, L^{d(d-1)})$ of the polynomial $Z^d + a_1 Z^{d-1} + \cdots + a_{d-1}Z + a_d \in S[Z]$ vanishes nowhere on Z then for any given differential form $g \in C^r_{p,q}(Y, M^k)$, $n \in \mathbb{Z}$, there exist unique differential forms $g_j \in C^r_{p,q}(Z, L^{k-j})$, $j = 0, \ldots d-1$, such that

$$g = \sum_{j=0}^{d-1} \pi^* g_j \otimes s^j . \tag{4.2}$$

Furthermore, the differential form g is $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact) if and only if all differential forms g_j, $j = 0, \ldots, d-1$, are $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact).

Proof. Since the claim is local with respect to Z, we may assume that $L = \mathcal{O}_Z$ and that π has d distinct right inverses $r_i : Z \to Y$, $qr_i = \mathrm{id}_Z$, $i = 1, \ldots, d$. Let $s_i = r_i^* s \in H^0(Z, \mathcal{O}_Z)$, $i = 1, \ldots, d$. Then $s_{i_1}(b) \neq s_{i_2}(b)$ for $i_1 \neq i_2$ and $b \in Z$ because D vanishes nowhere on Z. Eq. (4.2) is equivalent to the linear system

$$\sum_{j=0}^{d-1} s_i^j g_j = r_i^* g , \qquad i = 1, \ldots, d .$$

Its determinant is

$$\Delta = \prod_{1 \le i_1 < i_2 \le d} (s_{i_2} - s_{i_1}) .$$

Since $\Delta^2 = D$, the holomorphic function Δ vanishes nowhere on Z. Thus the differential forms g_j, $j = 1, \ldots, d$, are determined uniquely by Cramer's formulae: $g_j = \Delta^{-1} \det(A_j)$, $j = 0, \ldots, d-1$, where A_j, $j = 0, \ldots, d-1$, is the matrix

$$A_j = \begin{pmatrix} 1 & s_1 & \cdots & s_1^{j-1} & r_1^* g & s_1^{j+1} & \cdots & s_1^{d-1} \\ 1 & s_2 & \cdots & s_2^{j-1} & r_2^* g & s_2^{j+1} & \cdots & s_2^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & s_d & \cdots & s_d^{j-1} & r_d^* g & s_d^{j+1} & \cdots & s_d^{d-1} \end{pmatrix} .$$

Since $r_i^* g \in C^r_{p,q}(Z)$ for $i = 1, \ldots, d$, all differential forms g_j, $j = 0, \ldots, d-1$, also belong to $C^r_{p,q}(Z)$. We note that $\overline{\partial} g = \sum_{j=0}^{d-1} \pi^*(\overline{\partial} g_j) \otimes s^j$ because the section s is holomorphic. Since the representation (4.2) is unique and the homomorphism π^* is injective, the differential forms g_j, $j = 0, \ldots, d-1$, are $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact) if and only if the differential form g is $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact). □

Now we can deal with Representation (4.1).

Proposition 4.2. Let (V', W, z) be an admissible triple for X in $\boldsymbol{P}$. Then for any differential form $g \in C^r_{p,q}(X_D, \mathcal{O}_X(k))$, $k \in \mathbb{Z}$, there exist unique differential forms $g_j \in C^r_{p,q}(\boldsymbol{P}(V')_D, \mathcal{O}_{\boldsymbol{P}(V')}(k-j))$, $j = 0, \ldots, d-1$, such that

$$g = \sum_{j=0}^{d-1} (p|_{X_D})^* g_j \otimes z^j .$$

Furthermore, the differential form g is $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact) if and only if all differential forms g_j, $j = 0, \dots, d-1$, are $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact).

Proof. By Proposition 3.4, the holomorphic map $p|_{X_D} : X_D \to \boldsymbol{P}(V')_D$ is a finite covering of degree $d = \deg X$. Proposition 4.2 now follows from Proposition 4.1 with $Y = X_D$, $Z = \boldsymbol{P}(V')_D$, $L = \mathcal{O}_{\boldsymbol{P}(V')}(1)|_{\boldsymbol{P}(V')_D}$, and $s = z|_{X_D}$. □

We also need a version of Proposition 4.2 for an algebraic submanifold X of finite codimension n in a Banach space V. Let (V', W) be an admissible factorisation for X as in Section 2, and let $Z_1, \dots, Z_n$ be a basis of W^* such that $z = Z_1 + I(X) \in \mathbb{C}[X]$ generates the field of fractions of $\mathbb{C}[X]$ over the field of fractions of $\mathbb{C}[V']$. Let $D \in \mathbb{C}[V']$ be the discriminant of the minimal polynomial F of z over the field of fractions of $\mathbb{C}[V']$. By Proposition 2.3, the holomorphic map $p_D = p|_{X_D} : X_D \to V'_D$ is a covering of degree $d = \deg F$. We note that the vector bundle $T^{p,q}V' \to V'$ is canonically isomorphic to the trivial bundle $V' \times \wedge^p V' \wedge^q \overline{V'} \to V'$.

Proposition 4.3. For any differential form $g \in C^r_{p,q}(X_D)$, there exist unique differential forms $g_j \in C^r_{0,q}(V'_D)$, $j = 0, \dots, d-1$, such that

$$g = \sum_{j=0}^{d-1} z^j \, p_D^* \, g_j \, .$$

If U is an open subset of V'_D such that p_D has d distinct right inverses $r_i : U \to X_D$ on U, $\pi \circ r_i = \mathrm{id}_U$, $i = 1, \dots, d$, then

$$g_j(b, \xi, \overline{\xi}) = D(b)^{-1} \, \Delta(b) \, \det A_j(b, \xi, \overline{\xi}) \, , \qquad j = 0, \dots, d-1 \, ,$$

for $b \in U$, $\xi \in \wedge^p V'$, $\overline{\xi} \in \wedge^q \overline{W'}$, where

$$\Delta(b) = \prod_{1 \le i_1 < i_2 \le d} (z(r_{i_2}(b)) - z(r_{i_1}(b)))$$

and $A_j(b, \xi, \overline{\xi})$ is the $d \times d$ matrix

$$\begin{pmatrix} 1 & z(r_1(b)) & \cdots & z(r_1(b))^{j-1} & r_1^* g\,(b, \xi, \overline{\xi}) & z(r_1(b))^{j+1} & \cdots & z(r_1(b))^{d-1} \\ 1 & z(r_2(b)) & \cdots & z(r_2(b))^{j-1} & r_2^* g\,(b, \xi, \overline{\xi}) & z(r_2(b))^{j+1} & \cdots & z(r_2(b))^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z(r_d(b)) & \cdots & z(r_d(b))^{j-1} & r_d^* g\,(b, \xi, \overline{\xi}) & z(r_d(b))^{j+1} & \cdots & z(r_d(b))^{d-1} \end{pmatrix}.$$

Proof. We use Proposition 4.2 with $Y = X_D$, $Z = V'_D$, $L = \mathcal{O}_{V'_D}$, and $s = z$. Since $s_i = r_i^*(s) = z \circ r_i$, we have $s_i(b) = z(r_i(b))$, $i = 1, \dots, d$. Hence

$$\begin{aligned} \Delta(b)^2 &= \prod_{1 \le i_1 < i_2 \le d} (s_{i_2}(b) - s_{i_1}(b))^2 \\ &= \prod_{1 \le i_1 < i_2 \le d} (z(r_{i_2}(b)) - z(r_{i_1}(b)))^2 = D(b) \, , \end{aligned}$$

which gives $D(b)^{-1}\Delta(b) = \Delta(b)^{-1}$. The claim now follows from Cramer's formulae as in the proof of Proposition 4.2. $\square$

Propositions 4.2 and 4.3 are used in full generality in [3]. Here we use them only for $p = q = 0$. Let us consider first the affine case.

Lemma 4.4. We keep the assumptions and the notation of Proposition 4.3. Let $g \in H^0(X, \mathcal{O}_X)$ and $g_j \in H^0(V'_D, \mathcal{O}_{V'})$, $j = 0, \ldots, d-1$, be such that

$$g = \sum_{j=0}^{d-1} z^j \, p_D^* \, g_j \, .$$

Then there exist $\tilde{g}_j \in H^0(V', \mathcal{O}_{V'})$, $j = 0, \ldots, d-1$ such that $Dg_j = \tilde{g}_j|_{V'_D}$ for $j = 0, \ldots, d-1$.

Proof. By virtue of Riemann's removable singularity theorem it is enough to show that, for any $b_0 \in Z(D)$, there is a neighbourhood U of b_0 such that all functions Dg_j, $j = 0, \ldots, d-1$, are bounded on $U \cap V'_D$. Let $G_j : X^d \to \mathbb{C}$, $j = 0, \ldots, d-1$, be the holomorphic function given by

$$G_j(x_1, x_2, \ldots, x_d) = \Delta(x_1, x_2, \ldots, x_d) \det A_j(x_1, x_2, \ldots, x_d)$$

where

$$\Delta(x_1, x_2, \ldots, x_d) = \prod_{1 \le i_1 < i_2 \le d} (z(x_{i_2}) - z(x_{i_1}))$$

and $A_j(x_1, x_2, \ldots, x_d)$ is the matrix

$$\begin{pmatrix} 1 & z(x_1) & \cdots & z(x_1)^{j-1} & g(x_1) & z(x_1)^{j+1} & \cdots & z(x_1)^{d-1} \\ 1 & z(x_2) & \cdots & z(x_2)^{j-1} & g(x_2) & z(x_2)^{j+1} & \cdots & z(x_2)^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z(x_d) & \cdots & z(x_d)^{j-1} & g(x_d) & z(x_d)^{j+1} & \cdots & z(x_d)^{d-1} \end{pmatrix}.$$

Since the fiber $p^{-1}(b_0)$ is a finite set, there is a neighbourhood N of $p^{-1}(b_0)$ such that the functions G_j, $j = 0, \ldots, d-1$, are bounded on N^d and since $p : X \to V'$ is a proper map, there is a neighbourhood U of b_0 in V' such that $p^{-1}(U) \subset N$. By Proposition 4.3, $D(b)g_j(b) = G_j(x_1(b), x_2(b), \ldots, x_d(b))$, $j = 0, \ldots, d-1$, for $b \in V'_D$, where $\{x_1(b), x_2(b), \ldots, x_d(b)\} = p^{-1}(b)$. Hence all functions g_j, $j = 0, \ldots, d-1$, are bounded on $U \cap V'_D$ because $\{x_1(b)\} \times \{x_2(b)\} \times \ldots \times \{x_d(b)\} \in N^d$ for any $b \in U \cap V'_D$. $\square$

Lemma 4.5. We keep the assumptions and the notation of Proposition 4.2. Let $g \in H^0(X, \mathcal{O}_X(k))$, $k \in \mathbb{Z}$, and $g_j \in H^0(\boldsymbol{P}(V')_D, \mathcal{O}_{\boldsymbol{P}(V')}(k-j))$, $j = 0, \ldots, d-1$, be such that

$$g|_{X_D} = \sum_{j=0}^{d-1} (p|_{X_D})^* g_j \otimes z^j \, .$$

Then there exist $\tilde{g}_j \in H^0(\boldsymbol{P}(V'), \mathcal{O}_{\boldsymbol{P}(V')}(k - j + \deg D))$, $j = 0, \ldots, d-1$, such that $Dg_j = \tilde{g}_j|_{\boldsymbol{P}(V')_D}$ for $j = 0, \ldots, d-1$.

Proof. Let $s = \deg D = d(d-1)$. It is enough to show that for any $h \in V'^*$, $h \neq 0$, the holomorphic functions $Dg_j h^{j-k-s} \in H^0(\boldsymbol{P}(V')_D \cap \boldsymbol{P}(V')_h, \mathcal{O}_{\boldsymbol{P}(V')_h})$, $j = 0, \ldots, d-1$, can be extended to holomorphic functions on $\boldsymbol{P}(V')_h$. Since

$$gh^{-k}|_{X_D \cap X_h} = \sum_{j=0}^{d-1} (z/h)^j \, (p|_{X_D \cap X_h})^* (g_j h^{j-k}) \,,$$

this follows from Lemma 4.4. □

Proposition 4.6. If X is a submanifold of finite codimension in $\boldsymbol{P} = \boldsymbol{P}(V)$ and (V', W, z) is an admissible triple for X in $\boldsymbol{P}$, then for any $g \in H^0(X, \mathcal{O}_X(k))$, $k \in \mathbb{Z}$, there exists $\tilde{g} \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k + \deg D))$ such that $Dg|_{X_D} = \tilde{g}|_{X_D}$.

Proof. Let $g_j \in H^0(\boldsymbol{P}(V')_D, \mathcal{O}_{\boldsymbol{P}(V')})(k-j)$, $j = 0, \ldots, d-1$, and $\tilde{g}_j \in H^0(\boldsymbol{P}(V'), \mathcal{O}_{\boldsymbol{P}(V')}(k - j + \deg D))$, $j = 0, \ldots, d-1$, be as in Lemma 4.5. Choose a bounded linear functional $Z : V \to \mathbb{C}$ such that $Z|X = Z + I(X) = z$, and set $\tilde{g}_0 = \sum_{j=0}^{d-1} \pi^* \tilde{g}_j \otimes Z^j \in H^0(\boldsymbol{P} \setminus \boldsymbol{P}(W), \mathcal{O}_{\boldsymbol{P}}(k + \deg D))$, where π is the vector bundle $\boldsymbol{P} \setminus \boldsymbol{P}(W) \to \boldsymbol{P}(V')$. Since $\operatorname{codim}_{\boldsymbol{P}} \boldsymbol{P}(W) < 1$, there exists a $\tilde{g} \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k + \deg D))$ such that $\tilde{g}|_{\boldsymbol{P} \setminus \boldsymbol{P}(W)} = \tilde{g}_0$. According to Lemma 4.5, we have $\tilde{g}_0|_{X_D} = Dg|_{X_D}$ which implies $\tilde{g}|_{X_D} = Dg|_{X_D}$. □

5. A DOLBEAULT ISOMORPHISM FOR INFINITE-DIMENSIONAL PROJECTIVE SPACES

In this section, we will assume that V is a Banach space which admits smooth partitions of unity. In general we say that manifold X admits smooth partitions of unity if for any open cover $\{U_i\}_{i \in I}$ of X there are $\theta_i \in C^\infty(X)$, supported in U_i such that $\sum_{i \in I} \theta_i = 1$, the sum being locally finite. Hilbert spaces are examples of such manifolds. Separable and reflexive Banach spaces that localise are other examples. Paracompact manifolds modeled on spaces that admit smooth partitions of unity also admit smooth partitions of unity. We refer to [2] for more details. In particular if V is a Banach space that admits smooth partitions of unity, then the associated projective space $\boldsymbol{P} = \boldsymbol{P}(V)$ also admits smooth partitions of unity.

For a finite-dimensional complex manifold X, the Dolbeault cohomology groups and the Čech cohomology groups of a holomorphic vector bundle on X are the same by the Dolbeault isomorphism. Let X be a submanifold of finite codimension in $\boldsymbol{P}$ and let $E \to X$ be a holomorphic vector bundle over X. We will consider a covering $\{X_i\}_{i \in I}$ of X with Zariski open sets and define a complex $\mathcal{C}(X, E)$ which is a subcomplex of the usual Čech complex associated with $\{X_i\}_{i \in I}$ and E. In this section, we will prove that $H^q(\mathcal{C}(\boldsymbol{P}, E)) \cong H^{0,q}(\boldsymbol{P}, E)$ for $q \geq 0$. Since

$H^{0,q}(\boldsymbol{P}, E) = 0$ for $q \geq 1$ (see [4, Theorem 7.3]), we obtain $H^q(\mathcal{C}(\boldsymbol{P}, E)) = 0$ for $q \geq 1$. The vanishing of the higher cohomology groups of the complex $\mathcal{C}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n))$, $n \in \mathbb{Z}$, is used in the next section.

Let $\mathcal{D} = \{D_i \in C[V]\}_{i \in I}$ be a collection of homogeneous polynomials such that $\mathcal{U} = \{\boldsymbol{P}_{D_i}\}_{i \in I}$ is a covering of $\boldsymbol{P}$. The degree of the polynomial D_i will be denoted by d_i, $i \in I$. The open set

$$X_{D_{i_0}} \cap \ldots \cap X_{D_{i_q}} = \{x \in X \,:\, D_{i_m}(x) \neq 0 \text{ for } m = 0, \ldots, q\}$$

will be denoted by $X_{i_0 \ldots i_q}$. The complex $\mathcal{C}(X, E)$, corresponding to the collection $\mathcal{D}$, is defined in the following way. For any natural number N, we define first a complex of abelian groups

$$\mathcal{C}_N(X, E) = \{\mathcal{C}_N^q(X, E), \delta\}_{q \geq 0}$$

as follows: Let $\mathcal{C}_N^q(X, E)$ be the subgroup of $\prod_{i_0, \ldots, i_q \in I} H^0(X_{i_0 \ldots i_q}, E)$ that consists of all $\varphi = \{\varphi_{i_0 \ldots i_q} \in H^0(X_{i_0 \ldots i_q}, E)\}_{i_0, \ldots, i_q \in I}$ such that for any $i_0, \ldots, i_q \in I$ there exists a global section $\tilde{\varphi}_{i_0 \ldots i_q} \in H^0(X, E \otimes \mathcal{O}_X(Nd_{i_0} + \ldots + Nd_{i_q}))$ for which

$$\varphi_{i_0 \ldots i_q} = (D_{i_0} \cdots D_{i_q})^{-N} (\tilde{\varphi}_{i_0 \ldots i_q})|_{X_{i_0 \ldots i_q}}.$$

We use the standard convention for alternate cochains: if $\{m_0 \ldots m_q\}$ is a permutation of $\{0 \ldots q\}$, then $\varphi_{i_{m_0} \ldots i_{m_q}} = (-1)^{\epsilon(m_0 \ldots m_q)} \varphi_{i_0 \ldots i_q}$, where $\epsilon(m_0 \ldots m_q)$ is the parity of the permutation $\{m_0 \ldots m_q\}$. The differential $\delta : \mathcal{C}_N^q(X, E) \to \mathcal{C}_N^{q+1}(X, E)$ is the Čech coboundary operator:

$$(\delta\varphi)_{i_0 \ldots i_{q+1}} = \sum_{m=0}^{q+1} (-1)^m \, \varphi_{i_0 \ldots \widehat{i_m} \ldots i_{q+1}}|_{X_{i_0 \ldots i_{q+1}}}. \tag{5.1}$$

Since

$$(D_{i_0} \cdots D_{i_{q+1}})^N (\delta\varphi)_{i_0 \ldots i_{q+1}} = \left\{ \sum_{m=0}^{q+1} (-1)^m \, (D_{i_m}^N \, \tilde{\varphi}_{i_0 \ldots \widehat{i_m} \ldots i_{q+1}}) \right\} \Big|_{X_{i_0 \ldots i_{q+1}}}, \tag{5.2}$$

δ is well defined. We note that for any $N \in \mathbb{N}$, there is a natural injective chain map

$$\mathcal{C}_N(X, E) \to \mathcal{C}_{N+1}(X, E).$$

The complex $\mathcal{C}(X, E)$ is now defined as the union of all complexes $\mathcal{C}_N(X, E)$:

$$\mathcal{C}(X, E) = \bigcup_{N=0}^{\infty} \mathcal{C}_N(X, E). \tag{5.3}$$

Remark 1. The definition of the complex $\mathcal{C}(X, E)$ was suggested by the proof of [4, Theorem 8.2]. We note, however, that the proof of [4, Theorem 8.2] is not rigorous since it assumes implicitly that $\boldsymbol{P}$ is paracompact with the Zariski topology. We also note that the complex $\mathcal{C}(X,E)$ depends not only on the covering $\mathcal{U} = \{\boldsymbol{P}_{D_i}\}_{i\in I}$ but also on the collection of polynomials $\mathcal{D} = \{D_i\}_{i\in I}$. For example, let $I = \mathbb{N}$ and let $\mathcal{D} = \{D_i \in C[V]\}_{i\in I}$ be a collection of homogeneous polynomials such that $\mathcal{U} = \{\boldsymbol{P}_{D_i}\}_{i\in I}$ is a covering of $\boldsymbol{P}$. Let $D'_i = (D_i)^i$, $i \in I$. Then the covering $\mathcal{U}' = \{\boldsymbol{P}_{D'_i}\}_{i\in I}$ is the same as the covering $\mathcal{U}$, but the complex $\mathcal{C}'(X,E)$, corresponding to the collection of homogeneous polynomials $\mathcal{D}' = \{D'_i\}_{i\in I}$, is not necessarily the same as the complex $\mathcal{C}(X,E)$.

The next theorem is the main result of this section.

Theorem 5.1. Let V be a complex Banach space that admits smooth partitions of unity and let $\boldsymbol{P} = \boldsymbol{P}(V)$. Let $\{D_i\}_{i\in I}$ be a collection of homogeneous polynomials such that $\boldsymbol{P} = \cup_{i\in I}\boldsymbol{P}_{D_i}$. Then $H^q(\mathcal{C}(\boldsymbol{P},E)) = 0$, $q \geq 1$, for any holomorphic vector bundle $E \to \boldsymbol{P}$ of finite rank over $\boldsymbol{P}$.

To prove Theorem 5.1, it is enough to show that the group $H^q(\mathcal{C}_N(\boldsymbol{P},E)) = 0$ for $N \in \mathbb{N}$, $q \geq 1$. We will prove that $H^q(\mathcal{C}_N(\boldsymbol{P},E)) \cong H^{0,q}(\boldsymbol{P},E)$ for $N \in \mathbb{N}$, $q \geq 0$. To this end we define a double complex $\mathcal{B}_N(\boldsymbol{P},E) = \{\mathcal{B}_N^{pq}(\boldsymbol{P},E,\delta,\overline{\partial}\}_{p,q\geq 0}$ as follows. Let $\mathcal{B}_N^{pq}(\boldsymbol{P},E)$ be the subgroup of the group $\prod_{i_0,\dots,i_q\in I} C^\infty_{0,p}(\boldsymbol{P}_{i_0\dots i_q},E)$ that consists of all $\varphi \in \prod_{i_0,\dots,i_q\in I} C^\infty_{0,p}(\boldsymbol{P}_{i_0\dots i_q},E)$ such that for any $i_0,\dots,i_q \in I$ there exists a global section

$$\tilde{\varphi}_{i_0\dots i_q} \in C^\infty_{0,p}(\boldsymbol{P}, E\otimes\mathcal{O}_{\boldsymbol{P}}(Nd_{i_0}+\dots+Nd_{i_q}))$$

for which

$$\varphi_{i_0\dots i_q} = (D_{i_0}\cdots D_{i_q})^{-N}\varphi_{i_0\dots i_q}|_{\boldsymbol{P}_{i_0\dots i_q}}\,.$$

The differential

$$\delta : \mathcal{B}_N^{pq}(\boldsymbol{P},E) \to \mathcal{B}_N^{p\,q+1}(\boldsymbol{P},E)$$

is given by formula (5.1). Formula (5.2) shows that δ is well defined. The differential

$$\overline{\partial} : \mathcal{B}_N^{pq}(\boldsymbol{P},E) \to \mathcal{B}_N^{p+1\,q}(\boldsymbol{P},E)$$

is given by

$$(\overline{\partial}\varphi)_{i_0\dots i_q} = \overline{\partial}\varphi_{i_0\dots i_q}, \qquad i_0,\dots,i_q \in I\,,$$

for $\varphi \in \mathcal{B}_N^{pq}(\boldsymbol{P},E)$. Since

$$(D_{i_0}\cdots D_{i_q})^N(\overline{\partial}\varphi)_{i_0\dots i_q} = \overline{\partial}((D_{i_0}\cdots D_{i_q})^N\varphi_{i_0\dots i_q}) = \overline{\partial}(\tilde{\varphi}_{i_0\dots i_q}|_{\boldsymbol{P}_{i_0\dots i_q}}),$$

we see that $\overline{\partial}$ is well defined, too.

Lemma 5.2. Let V be a Banach space that admits smooth partitions of unity and let $\boldsymbol{P} = \boldsymbol{P}(V)$. Then, for any $n \in \mathbb{Z}$ and $N \in \mathbb{N}$:

i) $H^p_{\overline{\partial}}(\mathcal{B}^{\cdot q}_N(\boldsymbol{P},E)) = 0$ for $p \geq 1$;
ii) $H^q_\delta(\mathcal{B}^{p\cdot}_N(\boldsymbol{P},E)) = 0$ for $q \geq 1$.

Proof. Suppose $p \geq 1$, and let $\varphi \in \mathcal{B}^{pq}_N(\boldsymbol{P},E)$ be such that $\overline{\partial}\varphi = 0$. For $i_0,\dots,i_q \in I$, let $\tilde{\varphi}_{i_0\dots i_q} \in C^\infty_{0,p}(\boldsymbol{P}, E\otimes\mathcal{O}_{\boldsymbol{P}}(Nd_{i_0}+\dots+Nd_{i_q}))$ be the unique form such that

$$(D_{i_0}\cdots D_{i_q})^N\varphi_{i_0\dots i_q} = \tilde{\varphi}_{i_0\dots i_q}|_{\boldsymbol{P}_{i_0\dots i_q}}\,.$$

Since the form $\varphi_{i_0\dots i_q}$ is closed, the form $\tilde{\varphi}_{i_0\dots i_q}$ is closed, too. By [4, Theorem 7.3] there exists a form $\tilde{\psi}_{i_0\dots i_q} \in C^\infty_{0,p-1}(\boldsymbol{P}, E\otimes\mathcal{O}_{\boldsymbol{P}}(Nd_{i_0}+\dots+Nd_{i_q}))$ such that $\overline{\partial}\tilde{\psi}_{i_0\dots i_q} = \tilde{\varphi}_{i_0\dots i_q}$. Let $\psi \in \mathcal{B}^{p-1\,q}_N(\boldsymbol{P},E)$ be the cochain given by

$$\psi_{i_0\dots i_q} = (D_{i_0}\cdots D_{i_q})^{-N}(\tilde{\psi}_{i_0\dots i_q}|_{\boldsymbol{P}_{i_0\dots i_q}}) \in C^\infty_{0,p-1}(\boldsymbol{P}_{i_0\dots i_q},E),\ i_0,\dots,i_q\in I\,.$$

Since $\overline{\partial}\psi_{i_0\dots i_q} = \varphi_{i_0\dots i_q}$ for all $i_0,\dots i_q \in I$, we obtain $\overline{\partial}\psi = \varphi$. This proves (i).

Suppose $q \geq 1$, and let $\varphi \in \mathcal{B}^{pq}_N(\boldsymbol{P},E)$ be such that $\delta\varphi = 0$. Let $\{\theta_i\}$ be a smooth partition of unity which is subordinated to the open covering $\{\boldsymbol{P}_i\}_{i\in I}$, and let $\overline{\varphi} \in \prod\limits_{i_0,\dots,i_{q-1}\in I} C^\infty_{0,p}(\boldsymbol{P}_{i_0\dots i_{q-1}},E)$ be the cochain given by

$$\overline{\varphi}_{i_0\dots i_{q-1}} = \sum_{i\in I}\theta_i\varphi_{ii_0\dots i_{q-1}} \tag{5.4}$$

for $i_0,\dots,i_{q-1} \in I$. (On the right-hand side of (5.4) we use alternate cochains.) A simple calculation shows that $\delta\overline{\varphi} = \varphi$ (see [1, Proposition 8.5]). Let us verify that $\overline{\varphi} \in \mathcal{B}^{p\,q-1}(\boldsymbol{P},E)$.

$$(D_{i_0}\cdots D_{i_{q-1}})^N\overline{\varphi}_{i_0\dots i_{q-1}} =$$
$$= \left\{\sum_{i\in I}\theta_i\,(D_i|_{\boldsymbol{P}_i})^{-N}\,[(D_iD_{i_0}\cdots D_{i_{q-1}})^N\varphi_{ii_0\dots i_{q-1}}]\right\}\Big|_{V_{i_0\dots i_{q-1}}}$$
$$= \left\{\sum_{i\in I}\theta_i\,(D_i|_{\boldsymbol{P}_i})^{-N}\,\tilde{\varphi}_{ii_0\dots i_{q-1}}\right\}\Big|_{V_{i_0\dots i_{q-1}}}$$

Since $\theta_i(D_i|_{\boldsymbol{P}_i})^{-N} \in C^\infty(\boldsymbol{P},\mathcal{O}_{\boldsymbol{P}}(-Nd_i))$ and $\operatorname{supp}\theta_i(D_i|_{\boldsymbol{P}_i})^{-N} \subset \operatorname{supp}\theta_i$ for all $i \in I$, we obtain

$$\sum_{i\in I}\theta_i\,(D_i|_{\boldsymbol{P}_i})^{-N}\,\tilde{\varphi}_{ii_0\dots i_{q-1}} \ \in\ C^\infty_{0,p}(\boldsymbol{P},E\otimes\mathcal{O}_{\boldsymbol{P}}(Nd_{i_0}+\dots+Nd_{i_{q-1}}))\,.$$

Hence $\overline{\varphi} \in \mathcal{B}^{p\,q-1}(\boldsymbol{P},E)$. Since $\delta\overline{\varphi} = \varphi$, part (ii) has been proved. □

Remark 2. We note that we were able to prove part (ii) of Lemma 5.2 because the same N "worked" for all cochains $\varphi_{i_0\dots i_q}$, $i_0,\dots,i_q \in I$, (cf. Remark 1.)

Now we can give a proof of Theorem 5.1.

Proof of Theorem 5.1. As it has already been mentioned, it is enough to show that $H^q(\mathcal{C}_N(\boldsymbol{P}, E)) = 0$ for all $q \geq 1$, $n \in \mathbb{Z}$, and $N \in N$. It is well known (see for example [1, Proposition 8.8]) that if conditions (i) and (ii) of Lemma 5.1 hold for a double complex $B = \{\mathcal{B}^{pq}, d', d''\}_{p,q\geq 0}$, then the groups $H^q_{d'}(H^0_{d''}(B))$ and $H^q_{d''}(H^0_{d'}(B))$ are naturally isomorphic for all $q \geq 0$. We note that the complex $H^0_{\delta}(B(\boldsymbol{P}, E))$ is the Dolbeault complex of the vector bundle E on $\boldsymbol{P}$, and the complex $H^0_{\overline{\partial}}(B(\boldsymbol{P}, E))$ is just the complex $\mathcal{C}_N(\boldsymbol{P}, E)$. Since the higher Dolbeault cohomology groups of the vector bundle E on $\boldsymbol{P}$ vanish by [4, Theorem 7.3], we obtain $H^q(\mathcal{C}_N(\boldsymbol{P}, E)) = 0$ for $q \geq 1$, $N \in \mathbb{N}$. □

6. A DOLBEAULT ISOMORPHISM FOR COMPLETE INTERSECTIONS IN INFINITE-DIMENSIONAL PROJECTIVE SPACES

In this section we assume that X is a complete intersection in $\boldsymbol{P}$, and that $\{(V_i', W_i, z_i)\}_{i\in I}$ is a collection of admissible triples for X such that $\mathcal{U} = \{\boldsymbol{P}_{D_i}\}_{i\in I}$ is a covering of $\boldsymbol{P}$. According to Corollary 3.10, such collections exist for every submanifold X of finite codimension in $\boldsymbol{P}$. Let $\mathcal{C}(X, \mathcal{O}_X(k))$ be the complex (5.3) corresponding to the collection of homogeneous polynomials $\mathcal{D} = \{D_i\}_{i\in I}$ and to the line bundle $\mathcal{O}_X(k)$, $k \in \mathbb{Z}$. In this setup all polynomials D_i, $i \in I$, are of the same degree $d_i = d(d-1)$, where $d = \deg X$. We will show that if $\boldsymbol{P}$ admits smooth partitions of unity, then $H^q(\mathcal{C}(X, \mathcal{O}_X(k))) = 0$ for $q \geq 1$, $k \in \mathbb{Z}$.

Before dealing with the general case, let us outline the argument in the case of a hypersurface. Suppose X is the set of zeros of a homogeneous polynomial $P \in C[V]$ of degree d. Then multiplication by P yields the exact sequence of sheaves

$$0 \leftarrow \mathcal{O}_X(k) \leftarrow \mathcal{O}_{\boldsymbol{P}}(k) \xleftarrow{P} \mathcal{O}_{\boldsymbol{P}}(k-d) \leftarrow 0. \tag{6.1}$$

The sequence (6.1) induces an exact sequence of complexes

$$0 \leftarrow \mathcal{C}(X, \mathcal{O}_X(k)) \leftarrow \mathcal{C}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k)) \xleftarrow{P} \mathcal{C}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k-d)) \leftarrow 0 \tag{6.2}$$

Since by Theorem 5.1 $H^q(\mathcal{C}(P, \mathcal{O}_{\boldsymbol{P}}(k))) = 0$ for $q \geq 1$, $k \in \mathbb{Z}$, the long exact sequence of cohomology groups that is associated with the short exact sequence (6.2) yields $H^q(\mathcal{C}(X, \mathcal{O}_X(k))) = 0$ for $q \geq 1$, $k \in \mathbb{Z}$. This argument carries over to complete intersections in $\boldsymbol{P}$ because the exact sequence (6.1) is a special case of the Koszul complex.

To define complete intersections in $\boldsymbol{P}$, we need the notion of a regular sequence from commutative algebra.

Let A be a commutative ring and let $a_1, \ldots, a_n \in A$. Let $I_j, j = 1, \ldots, n$, be the ideal generated by $a_1, \ldots, a_j$ in A. The sequence $a_1, \ldots, a_j$ is called *regular* if $I_k \neq A$ and $a_j + I_{j-1}$ is not a zero divisor in the factor-ring A/I_{j-1} for $j = 1, \ldots, n$.

Given a commutative ring A and $a_1, \ldots, a_n \in A$, we define a complex K as follows:

$$K_0 \leftarrow \cdots \leftarrow K_{p-1} \xleftarrow{\mathrm{d}} K_p \leftarrow \cdots \leftarrow K_n \leftarrow 0$$

(cf. [5]) Set $K_0 = A$. For $1 \leq p \leq k$, let $K_p = \oplus A e_{j_1 \dots j_p}$ be the free A-module of rank $\binom{k}{p}$ with basis $\{e_{j_1 \dots j_p}\}_{1 \leq j_1 < \dots < j_p \leq k}$. The differential $\mathrm{d} : K_p \to K_{p-1}$ is given by

$$\mathrm{d}(e_{j_1 \dots j_p}) = \sum_{r=1}^{p} (-1)^{r-1} a_{j_r} e_{j_1 \dots \hat{j}_r \dots j_p} ;$$

(for $p = 1$, set $\mathrm{d}(e_j) = a_j$). One checks easily that $\mathrm{dd} = 0$. The complex K is called the Koszul complex corresponding to $a_1, \dots, a_n$. We note that $d(K_1) = I_n \subset A$ and $\mathrm{coker}\{K_0 \leftarrow K_1\} = A/I_n$.

The proof of the following important theorem can be found in [5].

Theorem 6.1 Let A be a commutative ring and let $a_1, \dots, a_n$ be a regular sequence in A. Then $H_p(K) = 0$ for $p > 0$. If A is an $\mathbb{N}$-graded ring and $a_1, \dots, a_n$ are homogeneous elements of positive degree, then the converse is also true.

It follows from Theorem 6.1 that if A is an $\mathbb{N}$-graded ring and $a_1, \dots, a_n$ is a regular sequence that consists of homogeneous elements of positive degree then any permutation of $a_1, \dots, a_n$ is also a regular sequence.

Definition. A submanifold X of finite codimension in $\boldsymbol{P}$ is called a *complete intersection* if there exists a regular sequence of homogeneous polynomials $P_1, \dots, P_n$ that generates the ideal $I(X)$.

From now on we assume that X is a complete intersection in $\boldsymbol{P}$, and $P_1, \dots, P_n$ is a given regular sequence of homogeneous polynomials in $C[V]$ that generates $I(X)$. We will denote by K the Koszul complex corresponding to $P_1, \dots, P_n$.

Let $\mathcal{C}(X)$, $\mathcal{C}(\boldsymbol{P})$, and $\mathcal{C}_N(\boldsymbol{P})$, $N \in \mathbb{N}$, be the complexes

$$\mathcal{C}(X) = \bigoplus_{k \in \mathbb{Z}} \mathcal{C}(X, \mathcal{O}_X(k)), \qquad \mathcal{C}(\boldsymbol{P}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{C}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k)),$$
$$\text{and} \quad \mathcal{C}_N(\boldsymbol{P}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{C}_N(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k)), \qquad N \in \mathbb{N}.$$

Let us note that $\mathcal{C}(\boldsymbol{P}) = \bigcup_{N \in \mathbb{N}} \mathcal{C}_N(\boldsymbol{P})$. We are going to construct a resolution of the complex $\mathcal{C}(X)$

$$0 \leftarrow \mathcal{C}(X) \overset{r}{\leftarrow} \mathcal{C}_0 \leftarrow \cdots \leftarrow \mathcal{C}_{p-1} \overset{\mathrm{d}'}{\longleftarrow} \mathcal{C}_p \leftarrow \cdots \leftarrow \mathcal{C}_n \leftarrow 0 \tag{6.3}$$

such that $H^q(\mathcal{C}_p) = 0$ for $q > 0$ and $p = 0, \dots, k$. This will immediately imply $H^q(\mathcal{C}(X, \mathcal{O}_X(k))) = 0$ for $q > 0$ and any $n \in \mathbb{Z}$.

In the construction of the resolution (6.3) we will use the existence of a natural C[V]-module structure on the complexes $\mathcal{C}_N(\boldsymbol{P})$, $N \in \mathbb{N}$, and $\mathcal{C}(\boldsymbol{P})$. To exhibit this module structure, we consider homogeneous polynomials of degree m as sections of the line bundle $\mathcal{O}_{\boldsymbol{P}}(m)$. If $P \in C[V]_m$ and $\varphi \in \mathcal{C}_N^q(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$, then it is easy to verify that the collection

$$(P\varphi)_{i_0 \dots i_q} = P\varphi_{i_0 \dots i_q} \in H^0(\boldsymbol{P}_{i_0 \dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k+m)) \tag{6.4}$$

is in $\mathcal{C}_N^q(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k+m))$. Thus the group $\mathcal{C}_N^q(\boldsymbol{P}) = \oplus_{k \in \mathbf{Z}} \mathcal{C}_N^q(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$, $q \geq 0$, has a structure of a $C[V]$-module that is given by (6.4). It follows from (5.1) and (6.4) that the coboundary operator $\delta : \mathcal{C}_N^q(\boldsymbol{P}) \to \mathcal{C}_N^{q+1}(\boldsymbol{P})$ is a homomorphism of $C[V]$-modules. Since $\mathcal{C}(\boldsymbol{P}) = \cup_{N \in \mathbb{N}} \mathcal{C}_N(\boldsymbol{P})$ and since the $C[V]$-module structures on $\mathcal{C}_N(\boldsymbol{P})$ and $\mathcal{C}_{N+1}(\boldsymbol{P})$ agree for all $N \in \mathbb{N}$, the complex $\mathcal{C}(\boldsymbol{P})$ also has a $C[V]$-module structure.

Remark 1. For $N \in \mathbb{N}$ and $i_0, \dots, i_q \in I$, let $C[V](D_{i_0} \cdots D_{i_q})^{-N}$ be the $C[V]$-module generated by $(D_{i_0} \cdots D_{i_q})^{-N}$ in the field of fractions of $C[V]$. It is clear that $C[V](D_{i_0} \cdots D_{i_q})^{-N}$ is a free $C[V]$-module of rank 1. It follows from the definition of the groups $\mathcal{C}_N^q(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$ that, for any $\varphi \in \mathcal{C}_N^q(\boldsymbol{P}) = \oplus_{k \in \mathbf{Z}} \mathcal{C}_N^q(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$, there exist unique $\tilde{\varphi}_{i_0 \dots i_q} \in \oplus_{k \in \mathbf{Z}} H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$, $i_0, \dots, i_q \in I$ such that

$$\varphi_{i_0 \dots i_q} = (D_{i_0} \cdots D_{i_q})^{-N} (\tilde{\varphi}_{i_0 \dots i_q})|_{\boldsymbol{P}_{i_0 \dots i_q}}$$

for any $i_0, \dots, i_q \in I$. Since $\oplus_{k \in \mathbf{Z}} H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k)) = C[V]$, there exists an isomorphism of $C[V]$-modules

$$\mathcal{C}_N^q(\boldsymbol{P}) \cong \prod_{i_0, \dots, i_q \in I} C[V](D_{i_0} \cdots D_{i_q})^{-N} . \tag{6.5}$$

The resolution (6.3) is constructed as follows: for $p = 0, \dots, n$, let $\mathcal{C}_p$ and $\mathcal{C}_{p,N}$, $N \in \mathbb{N}$, be the complexes

$$\mathcal{C}_p = K_p \otimes_{C[V]} \mathcal{C}(\boldsymbol{P}) \quad \text{and} \quad \mathcal{C}_{p,N} = K_p \otimes_{C[V]} \mathcal{C}_N(\boldsymbol{P}), \quad N \in \mathbb{N} . \tag{6.6}$$

The differential $\mathrm{d} : K_p \to K_{p-1}$ induces chain maps

$$\mathrm{d}' = d \otimes \mathrm{id} : \mathcal{C}_p \to \mathcal{C}_{p-1} \quad \text{and} \quad \mathrm{d}' = d \otimes \mathrm{id} : \mathcal{C}_{p,N} \to \mathcal{C}_{p-1,N}, \quad N \in \mathbb{N} , \tag{6.7}$$

for $p = 1, \dots, k$.

Proposition 6.1. The sequence of complexes

$$\mathcal{C}_0 \leftarrow \cdots \leftarrow \mathcal{C}_{p-1} \stackrel{\mathrm{d}'}{\leftarrow} \mathcal{C}_p \leftarrow \cdots \leftarrow \mathcal{C}_n \leftarrow 0$$

is exact.

Proof. Since $\mathcal{C}(\boldsymbol{P}) = \cup_{N \in \mathbb{N}} \mathcal{C}_N(\boldsymbol{P})$ it is enough to check that for any $N \in \mathbb{N}$ and any $q \geq 0$ the sequence of $C[V]$-modules

$$\mathcal{C}_{0,N}^q \leftarrow \cdots \leftarrow \mathcal{C}_{p-1,N}^q \stackrel{\mathrm{d}'}{\leftarrow} \mathcal{C}_{p,N}^q \leftarrow \cdots \leftarrow \mathcal{C}_{n,N}^q \leftarrow 0 \tag{6.8}$$

is exact. It follows from (6.6), (6.6) and (6.7) that the complex (6.8) is isomorphic to the complex

$$K \otimes \mathcal{C}^q_N(\boldsymbol{P}) = K \otimes \prod_{i_0,\dots,i_q \in I} C[V](D_{i_0} \cdots D_{i_q})^{-N}.$$

Since each K_p, $p = 1, \dots, k$, is a *finitely generated* $C[V]$-module, there is an isomorphism of complexes

$$K \otimes \prod_{i_0,\dots,i_q \in I} C[V](D_{i_0} \cdots D_{i_q})^{-N} \cong \prod_{i_0,\dots,i_q \in I} K \otimes C[V](D_{i_0} \cdots D_{i_q})^{-N}. \quad (6.9)$$

Now we note that the complex on the right-hand side of (6.9) is exact because the Koszul complex is exact by Theorem 5.1, and $C[V](D_{i_0} \cdots D_{i_q})^{-N}$ is a free $C[V]$-module for any $i_0, \dots, i_q \in I$. □

It remains to define a surjective chain map $r : \mathcal{C}_0 \to \mathcal{C}(X)$ such that $\ker r = \operatorname{im}\{\mathcal{C}_1 \xrightarrow{d'} \mathcal{C}_0\}$. We note that the complex $\mathcal{C}_0$ coincides with the complex $\mathcal{C}(\boldsymbol{P})$ because $K_0 = C[V]$. Then the map $r : \mathcal{C}_0 \to \mathcal{C}(X)$ is given by the restriction of the sections of the line bundles $\mathcal{O}_{\boldsymbol{P}}(k)$, $k \in \mathbb{Z}$, to the submanifold X. More precisely, the restriction homomorphisms

$$\{H^0(\boldsymbol{P}_{i_0\dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k)) \to H^0(X_{i_0\dots i_q}, \mathcal{O}_X(k))\}_{i_0,\dots,i_q \in I},$$
$$H^0(\boldsymbol{P}_{i_0\dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k)) \ni \varphi_{i_0\dots i_q} \mapsto (\varphi_{i_0\dots i_q})|_{X_{i_0\dots i_q}} \in H^0(X_{i_0\dots i_q}, \mathcal{O}_X(k))$$

induce chain maps $r_N : \mathcal{C}_N(P, \mathcal{O}_{\boldsymbol{P}}(k)) \to \mathcal{C}_N(X, \mathcal{O}_X(k))$ for $k \in \mathbb{Z}$, $N \in \mathbb{N}$. The collection of chain maps $\{r_N\}_{N \in \mathbb{N}}$ then induces a natural chain map

$$r_k : \mathcal{C}(P, \mathcal{O}_{\boldsymbol{P}}(k)) \to \mathcal{C}(X, \mathcal{O}_X(k)) \quad (6.10)$$

for any $k \in \mathbb{Z}$. The next lemma is instrumental in the proof of the surjectivity of the chain map (6.10).

Lemma 6.2. Suppose that

$$\Phi \in H^0(X_{i_0\dots i_q}, \mathcal{O}_X(k)) \quad \text{and} \quad \widetilde{\Phi} \in H^0(X, \mathcal{O}_X(k + Nd_{i_0} + \cdots + Nd_{i_q}))$$

are such that $\Phi \doteq (D_{i_0} \cdots D_{i_q})^{-N}(\widetilde{\Phi}|_{X_{i_0\dots i_q}})$ for some $N \in \mathbb{N}$. Then there exist sections

$$\varphi \in H^0(\boldsymbol{P}_{i_0\dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k)) \text{ and } \tilde{\varphi} \in H^0(P, \mathcal{O}_{\boldsymbol{P}}(k + (N+1)d_{i_0} + \cdots + (N+1)d_{i_q}))$$

such that $\Phi = \varphi|_{X_{i_0\dots i_q}}$ and $\varphi = (D_{i_0} \cdots D_{i_q})^{-N-1}\tilde{\varphi}|_{\boldsymbol{P}_{i_0\dots i_q}}$.

Proof. By Proposition 4.6, there exists a homogeneous polynomial $P \in C[V]$ such that $\deg P = k + Nd_{i_0} + \cdots + Nd_{i_q} + d_{i_0}$ and $(D_{i_0}\widetilde{\Phi})|_{X_{i_0}} P|_{X_{i_0}}$. Let

$$\tilde{\varphi} = D_{i_1} \cdots D_{i_q} P \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k + (N+1)d_{i_0} + \cdots + (N+1)d_{i_q}))$$

and

$$\varphi = (D_{i_0} \cdots D_{i_q})^{-N-1} \tilde{\varphi}|_{\boldsymbol{P}_{i_0 \dots i_q}} \in H^0(\boldsymbol{P}_{i_0 \dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k)) .$$

Then $\Phi = \varphi|_{X_{i_0 \dots i_q}}$ which proves the lemma. □

Lemma 6.3. The natural chain map

$$r_k : \mathcal{C}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k)) \to \mathcal{C}(X, \mathcal{O}_X(k))$$

is surjective for any $k \in \mathbb{Z}$.

Proof. Let Φ be a cochain in $\mathcal{C}^q_N(X, \mathcal{O}_X(k))$. It follows from Lemma 6.2 that for any $i_0, \dots, i_q \in I$ there are $\tilde{\varphi}_{i_0 \dots i_q} \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k+(N+1)d_{i_0}+\cdots+(N+1)d_{i_q}))$ and $\varphi_{i_0 \dots i_q} \in H^0(\boldsymbol{P}_{i_0 \dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k))$ such that

$$\varphi_{i_0 \dots i_q}|_{X_{i_0 \dots i_q}} = \Phi_{i_0 \dots i_q} \qquad \text{and} \qquad \varphi_{i_0 \dots i_q} = (D_{i_0} \cdots D_{i_q})^{-N-1} (\tilde{\varphi}_{i_0 \dots i_q}|_{\boldsymbol{P}_{i_0 \dots i_q}}) .$$

By the definition of the group $\mathcal{C}^q_{N+1}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$, the cochain $\varphi = \{\varphi_{i_0 \dots i_q}\}_{i_0 \dots i_q \in I}$ belongs to $\mathcal{C}^q_{N+1}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$ and $r_k(\varphi) = \Phi$. □

Let $r : \mathcal{C}_0 = \mathcal{C}(\boldsymbol{P}) \to \mathcal{C}(X)$ be the chain map

$$r = \oplus_{k \in \mathbb{Z}} r_k \ : \ \mathcal{C}_0 = \oplus_{k \in \mathbb{Z}} \mathcal{C}(P, \mathcal{O}_{\boldsymbol{P}}(k)) \to \oplus_{k \in \mathbb{Z}} \mathcal{C}(X, \mathcal{O}_X(k)) = \mathcal{C}(X) ,$$

where $r_k : \mathcal{C}(P, \mathcal{O}_{\boldsymbol{P}}(k)) \to \mathcal{C}(X, \mathcal{O}_X(k))$, $k \in \mathbb{Z}$, is the chain map (6.10).

Lemma 6.4. The sequence of complexes $0 \leftarrow \mathcal{C}(X) \xleftarrow{r} \mathcal{C}_0 \xleftarrow{\mathrm{d}'} \mathcal{C}_1$ is exact.

Proof. By Lemma 6.3 the chain map r is surjective. It remains to show that $\ker r = \operatorname{im} \mathrm{d}'$. Let

$$\varphi = \{\varphi_{i_0 \dots i_q} \in \oplus_{k \in \mathbb{Z}} H^0(\boldsymbol{P}_{i_0 \dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k))\}_{i_0, \dots, i_q \in I} \in \mathcal{C}^q_{0,N}(\boldsymbol{P}) = \mathcal{C}^q_N(\boldsymbol{P})$$

be such that $r(\varphi) = 0$. For any $i_0, \dots, i_q \in I$, there exists a polynomial $\tilde{\varphi}_{i_0 \dots i_q} \in C[V]$ such that $\varphi_{i_0 \dots i_q} = (D_{i_0} \cdots D_{i_q})^{-N} (\tilde{\varphi}_{i_0 \dots i_q}|_{X_{i_0 \dots i_q}})$. Each polynomial $\tilde{\varphi}_{i_0 \dots i_q}$, $i_0, \dots, i_q \in I$, vanishes on X because $\varphi_{i_0 \dots i_q}$ vanishes on $X_{i_0 \dots i_q}$ and the set $X_{i_0 \dots i_q}$ is dense in X. Since the ideal $I(X)$ is generated by $P_1, \dots, P_n$, for any polynomial $\tilde{\varphi}_{i_0 \dots i_q}$, $i_0, \dots, i_q \in I$, there exist polynomials $P_{1, i_0 \dots i_q}, \dots, P_{n, i_0 \dots i_q}$, such that

$$\tilde{\varphi}_{i_0 \dots i_q} = \sum_{j=1}^{n} P_j P_{j, i_0 \dots i_q} .$$

Let $\varphi_j = \{(D_{i_0} \cdots D_{i_q})^{-N} P_{j, i_0 \dots i_q}\}_{i_0, \dots, i_q \in I} \in \mathcal{C}^q_N(\boldsymbol{P})$, $j = 1, \dots, n$. Then $\varphi = \sum_{j=1}^{k} P_j \varphi_j$. Set

$$\overline{\varphi} = \sum_{j=1}^{k} e_j \otimes \varphi_j \ \in \ K_1 \otimes \mathcal{C}^q_N(\boldsymbol{P}) = \mathcal{C}^q_{1,N}(\boldsymbol{P}) .$$

Then

$$\mathrm{d}'(\overline{\varphi}) = \mathrm{d}'\left(\sum_{j=1}^{n} e_j \otimes \varphi_j\right) = \sum_{j=1}^{n} \mathrm{d}'(e_j) \otimes \varphi_j = \sum_{j=1}^{n} P_j \otimes \varphi_j = \varphi .$$

Thus for any $N \in \mathbb{N}$, any $q \geq 0$, and any $\varphi \in \mathcal{C}^q_{0,N}(\boldsymbol{P})$ such that $r(\varphi) = 0$, there exists $\overline{\varphi} \in \mathcal{C}^q_{1,N}(\boldsymbol{P})$ such that $\mathrm{d}'(\overline{\varphi}) = \varphi$. □

Theorem 6.5. Let V be an infinite-dimensional Banach space that admits smooth partitions of unity and let $\boldsymbol{P} = \boldsymbol{P}(V)$. If X is a complete intersection in $\boldsymbol{P}$. then $H^q(\mathcal{C}(X, \mathcal{O}_X(k))) = 0$ for $q \geq 1$ and any $k \in \mathbb{Z}$.

Proof. The sequence of complexes

$$0 \leftarrow \mathcal{C}(X) \xleftarrow{r} \mathcal{C}_0 \leftarrow \cdots \leftarrow \mathcal{C}_{p-1} \xleftarrow{\mathrm{d}'} \mathcal{C}_p \leftarrow \cdots \leftarrow \mathcal{C}_n \leftarrow 0$$

is exact by Proposition 6.1 and Lemma 6.4. Since each K_p is a free $C[V]$-module, we have $H^q(\mathcal{C}_p) = K_p \otimes H^q(\mathcal{C}(\boldsymbol{P}))$ for $p = 0, \ldots, k$, and $q \geq 0$. By Theorem 5.1, $H^q(\mathcal{C}(\boldsymbol{P})) = 0$ for $q \geq 1$. Hence $H^q(\mathcal{C}_p) = 0$ for $p = 0, \ldots, n$ and $q \geq 1$. Let B_j be the complex $\operatorname{coker}\{\mathcal{C}_j \xleftarrow{\mathrm{d}'} \mathcal{C}_{j+1}\}$, $j = 0, \ldots, n$. We note that $B_0 = \mathcal{C}(X)$ and $B_n = \mathcal{C}_n$. For any $j = 1, \ldots, n$, we have a short exact sequence

$$0 \leftarrow B_{j-1} \xleftarrow{\mathrm{d}'} \mathcal{C}_j \leftarrow B_j \leftarrow 0$$

Using the long exact sequence of cohomology groups, we derive by descending induction on j that $H^q(B_{j-1}) = 0$ for $q \geq 1$ and $j = n, \ldots, 1$. Hence $H^q(\mathcal{C}(X)) = 0$ for $q \geq 1$ and this implies $H^q(\mathcal{C}(X, \mathcal{O}_X(k))) = 0$ for $q \geq 1$ and any $k \in \mathbb{Z}$. □

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: bkotzev@fmi.uni-sofia.bg