ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 98

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 98

A JUMP INVERSION THEOREM FOR THE INFINITE ENUMERATION JUMP

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In this paper we study partial regular enumerations for arbitrary recursive ordinal. We use the technique to obtain a jump inversion and omitting theorem for the infinite enumeration jump for the case of partial degrees.

Keywords: enumeration reducibility, enumeration jump, enumeration degrees, forcing
2000 MSC: 03D30

1. INTRODUCTION

In [2] Soskov introduces the notion of regular enumerations. Using them he proves the following jump inversion theorem:

Theorem (Soskov). *Let $k > n \geq 0$ and $B_0, \dots, B_k$ be arbitrary sets of natural numbers. Let $A \subseteq N$ and Q be a total set such that $\mathcal{P}(B_0, \dots, B_k) \leq_e Q$ and $A^+ \leq_e Q$. Suppose also that $A \not\leq_e \mathcal{P}(B_0, \dots, B_n)$. Then there exists a total set F having the following properties:*

(i) *For all $i \leq k$, $B_i \in \Sigma^F_{i+1}$;*

(ii) *For all i $1 \leq i \leq k$, $F^{(i)} \equiv_e F \oplus \mathcal{P}(B_0, \dots, B_{i-1})'$;*

(iii) *$F^{(k)} \equiv_e Q$;*

(iv) *$A \not\leq_e F^{(n)}$.*

Here $\mathcal{P}(B_0, \dots)$ is the polynomial set obtained from $B_0, B_1, \dots$ as defined in Section 2.

In [1] Soskov and Baleva generalize the notion of regular enumeration and obtain the following result for the infinite case:

Theorem (Soskov, Baleva) *Let $\{B_\alpha\}_{\alpha\leq\zeta}$ be a sequence of sets of natural numbers. Let $\{A_\gamma\}_{\gamma<\zeta}$ also be a sequence of sets of natural numbers, such that for all $\gamma<\zeta$ is true that $A_\gamma \not\leq_e \mathcal{P}_\gamma$. Finally, let Q be a total set such that $\mathcal{P}_\zeta \leq_e Q$ and $\bigoplus_{\gamma<\zeta} A_\gamma^+ \leq_e Q$. Then there is a total set F such that:*

(1) *For all $\gamma\leq\zeta$ it is true that $B_\gamma \leq_e F^{(\gamma)}$ uniformly in γ;*

(2) *For all $\gamma\leq\zeta$, if $\gamma=\beta+1$ then $F^{(\gamma)} \equiv_e F\oplus\mathcal{P}'_\beta$ uniformly in γ;*

(3) *For all limit $\gamma\leq\zeta$ it is true that $F^{(\gamma)}\equiv_e F\oplus\mathcal{P}_{<\gamma}$ uniformly in γ;*

(4) *$F^{(\zeta)}\equiv_e Q$;*

(5) *For all $\gamma<\zeta$ it is true that $A_\gamma\not\leq_e F^{(\gamma)}$.*

In this paper we will prove that this result also holds if we want the target set F to be partial, i.e., the degree $\mathbf{d}_e(F)$ to be partial. Namely, we will prove the following theorem:

Theorem 1.1. *Let $\{B_\alpha\}_{\alpha\leq\zeta}$ be a sequence of sets of natural numbers. Let also $\{A_\gamma\}_{\gamma<\zeta}$ be a sequence of sets of natural numbers, such that for all $\gamma<\zeta$ it is true that $A_\gamma\not\leq_e \mathcal{P}_\gamma$. Finally let Q be a total set such that $\mathcal{P}_\zeta\leq_e Q$ and $\bigoplus_{\gamma<\zeta}A_\gamma^+\leq_e Q$. Then there exists a set F such that $\mathbf{d}_e(F)$ is partial and:*

(1) *For all $\gamma\leq\zeta$ it is true that $B_\gamma\leq_e F^{(\gamma)}$ uniformly in γ;*

(2) *For all $\gamma\leq\zeta$, if $\gamma=\beta+1$ then $F^{(\gamma)}\equiv_e F^+\oplus\mathcal{P}'_\beta$ uniformly in γ;*

(3) *For all limit ordinals $\gamma\leq\zeta$ it is true that $F^{(\gamma)}\equiv_e F^+\oplus\mathcal{P}_{<\gamma}$ uniformly in γ;*

(4) *$F^{(\zeta)}\equiv_e Q$;*

(5) *For all $\gamma<\zeta$ it is true that $A_\gamma\not\leq_e F^{(\gamma)}$;*

(6) *F is quasiminimal over B_0, i.e. for all total sets X if $X\leq_e F$ then $X\leq_e B_0$.*

2. PRELIMINARIES

Let $W_0,\dots,W_i,\dots$ be the Gödel enumeration of the r.e. sets. We define the enumeration operator Γ_i for arbitrary set of natural numbers by $\Gamma_i(A)=\{x \mid (\exists\langle x,u\rangle\in W_i)(D_u\subseteq A)\}$, where D_u is the finite set with canonical code u. We define the relation $\leq_e$ over the sets of natural numbers by

$$A\leq_e B \iff \exists i(A=\Gamma_i(B)).$$

The relation $\leq_e$ is reflexive and transitive and defines a equivalence relation $\equiv_e$. We call the equivalence classes of $\equiv_e$ enumeration degrees.

The composition of two enumeration operators is also an enumeration operator. Beside this the index of the resulting operator is obtained uniformly from the indexes of the other ones. This means that there exists a recursive function $\mathfrak{c}$ such that $\Gamma_i(\Gamma_j(A))=\Gamma_{\mathfrak{c}(i,j)}(A)$ for arbitrary set A.

We define the "join" operator $\oplus$ by $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$. We set $A^+ = A \oplus \overline{A}$. We say that a set A of natural numbers is total iff $A \equiv_e A^+$. We say that the enumeration degree $\mathbf{a}$ is total iff there is a total set $A \in \mathbf{A}$. Otherwise we say that the enumeration degree is partial.

We define the enumeration jump to be $A' = L_A^+$, where $L_A = \{\langle x, i\rangle \mid x \in \Gamma_i(A)\}$. Using ordinal notation we can define the infinite enumeration jump. More precisely:

Let η be a recursive ordinal and let us fix an ordinal notation $e \in \mathcal{O}$ for η. For every ordinal $\alpha < \eta$ we will use the corresponding notation which is $<_{\mathcal{O}}$ then e (for an introduction on ordinal notations see [3]). Then, not distinguishing the ordinal from its notation, we define the α jump for $\alpha < \eta$ by means of transfinite induction:

(1) $A^{(0)} = A$
(2) If $\alpha = \beta + 1$ then $A^{(\alpha)} = (A^{(\beta)})'$
(3) If $\alpha = \lim(\alpha(p))$ then $A^{(\alpha)} = \{\langle p, x\rangle \mid x \in A^{(\alpha(p))}\}$.

Naturally the definition depends on the choice of the ordinal notation of α. Despite this, we can prove that if α_1 and α_2 are two different notations of α, then $A^{(\alpha_1)} \equiv_e A^{(\alpha_2)}$ (see [1], [3]), as in the case of the turing infinite jump.

We define the "polynomials" $\mathcal{P}_\alpha$ of the sets $B_0, \ldots, B_\alpha, \ldots$ with

Definition 2.1. Let ζ be a recursive ordinal and let $\{B_\alpha\}_{\alpha \le \zeta}$ be a sequence of sets of natural numbers. Then we define using transfinite induction the sets $\mathcal{P}_\alpha$ in the following way:

(1) $\mathcal{P}_0 = B_0$
(2) if $\alpha = \beta + 1$ then $\mathcal{P}_\alpha = \mathcal{P}'_\beta \oplus B_\alpha$;
(3) if $\alpha = \lim(\alpha(p))$ then $\mathcal{P}_\alpha = \mathcal{P}_{<\alpha} \oplus B_\alpha$, where

$$\mathcal{P}_{<\alpha} = \{\langle p, x\rangle \mid x \in \mathcal{P}_{\alpha(p)}\}$$

We also introduce the following notation:
For an arbitrary sequence of sets $\{C_\alpha\}_{\alpha<\zeta}$ we define the set $\bigoplus_{\alpha<\zeta} C_\alpha$ to be

$$\bigoplus_{\alpha<\zeta} C_\alpha = \{\langle \alpha, x\rangle \mid x \in C_\alpha\}.$$

We will consider partial functions $f : \mathbf{N} \multimap \mathbf{N}$. We will say that $f \le_e A$ iff $\langle f\rangle \le_e A$, where $\langle f\rangle$ is the graphic of f. We will use "partial" finite parts τ for which $\tau : [0, 2q+1] \longrightarrow \mathbf{N} \cup \{\perp\}$. We define the graphic of τ to be $\langle\tau\rangle = \{\langle x, y\rangle \mid x \le 2q+1 \;\&\; \tau(x) = y \ne \perp\}$ and we say that $\tau \subseteq f$ iff $\langle\tau\rangle \subseteq \langle f\rangle$. We define $\mathrm{lh}(\tau) = 2q+2$

We will assume that an effective and reversible coding of all finite sequences is fixed. Thus we have an effective and reversible coding for all finite parts. As usual from now on we will make no difference between a finite part and its code. Even more: we say that $\tau \leq \rho$ iff the inequality holds for the codes of the finite parts ρ and τ. By $\tau \subseteq \rho$ we will mean the usual extension property.

Finally we will say that the statement $\exists i P(i, x_1, \dots, x_n, A_1, \dots, A_k)$, where $i, x_1, \dots, x_n \in \mathbf{N}$ and $A_1, \dots, A_n \subseteq \mathbf{N}$, is uniformly true in $x_1, \dots, x_n$ for all $A_1, \dots A_k$ iff there exists a recursive function $h(x_1, \dots, x_n)$ such that for every $x_1, \dots, x_n \in \mathbf{N}$ and every $A_1, \dots, A_k \subseteq \mathbf{N}$ the statement

$$P(h(x_1, \dots, x_n), x_1, \dots, x_n, A_1, \dots, A_k)$$

is true.

Of course the construction of h is quite difficult and uninformative. Hence, when we have to prove that some statement is uniformly true, usually we will show a construction in which all the choices we have to make will be effective.

3. REGULAR ENUMERATIONS

The proof of the theorem in most of its parts repeats the proof of Soskov, Baleva theorem. A complete proof of the last one can be found in [1].

Let us first fix a recursive ordinal ζ and a sequence of sets $\{B_\alpha\}_{\alpha \leq \zeta}$.

The following definitions of ordinal approximation and predecessor as the proofs of their basic properties are due to Soskov and Baleva.

Definition 3.1. Let α be a recursive ordinal. We will say that $\overline{\alpha}$ is an approximation of α, iff $\overline{\alpha}$ is finite sequence of ordinals $\overline{\alpha} = \langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha \rangle$, where $\alpha_0 = 0$, $\alpha_0 < \alpha_1 < \dots < \alpha_n < \alpha$ and $n \geq -1$.

Definition 3.2. Let α be a recursive ordinal and let $\beta < \alpha$. Let also $\overline{\alpha} = \langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha \rangle$ is an approximation of α. We define recursively the notion of β-predecessor of $\overline{\alpha}$:

a) if $\beta = \alpha_i$ for some $0 \leq i \leq n$ then set $\overline{\beta} = \langle \alpha_0, \alpha_1, \dots, \alpha_i \rangle$;
b) if $\alpha_i < \beta < \alpha_{i+1}$ for some $0 \leq i < n$ then set $\overline{\beta}$ to be the β-predecessor of $\langle \alpha_0, \alpha_1, \dots, \alpha_{i+1} \rangle$;
c) if $\alpha_n < \beta < \alpha$ then

1) if $\alpha = \delta + 1$ and $\beta = \delta$ set $\overline{\beta} = \langle \alpha_0, \alpha_1, \dots, \alpha_n, \beta \rangle$;

2) if $\alpha = \delta + 1$ and $\beta < \delta$ then set $\overline{\beta}$ to be the β-predecessor of $\langle \alpha_0, \alpha_1, \dots, \alpha_n, \delta \rangle$;

3) if $\alpha = \lim \alpha(p)$, $p_0 = \mu p[\alpha(p) > \alpha_n]$ and $p_1 = \mu p[\alpha(p) > \beta]$ set $\overline{\beta}$ to be the β-predecessor of $\langle \alpha_0, \alpha_1, \dots \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \dots, \alpha(p_1) \rangle$.

The following lemmas give the basic properties of the ordinal approximation and predecessor. The full proofs can be found in [1].

Lemma 3.1. *For every ordinal approximation $\overline{\alpha}$ and every $\beta < \alpha$ there is a unique β-predecessor $\overline{\beta}$ of α.*

Lemma 3.2. *Let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha \rangle$ be an approximation of α. Then:*

(1) *If $\beta \leq \alpha_i$ for some $0 \leq i \leq n$ then $\overline{\beta} \preceq \overline{\alpha} \Leftrightarrow \overline{\beta} \preceq \overline{\alpha}_i$*

(2) *If for some $0 \leq i \leq n$, $\alpha_i \leq \beta < \alpha$ and $\langle \beta_0, \beta_1, \ldots, \beta_k \rangle$ is the β-predecessor of $\overline{\alpha}$ then $i < k$ and $\alpha_l = \beta_l$ for all $l = 0, \ldots, i$*

(3) *Let $\alpha = \delta + 1$, $\alpha_n < \delta$ and $\beta \leq \delta$. Then $\overline{\beta} \preceq \overline{\alpha} \Leftrightarrow \overline{\beta} \preceq \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \delta \rangle$*

(4) *Let $\alpha = \lim \alpha(p)$ be a limit ordinal and let $p_0 = \mu p[\alpha_n < \alpha(p)]$. Let also $p_1 \geq p_0$ be such that $\beta \leq \alpha(p_1)$. Then*

$$\overline{\beta} \preceq \overline{\alpha} \Leftrightarrow \overline{\beta} \preceq \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \ldots, \alpha(p_1) \rangle$$

Lemma 3.3. *Let $\gamma < \beta < \alpha$ be ordinals, $\overline{\gamma} \preceq \overline{\beta}$ and $\overline{\beta} \preceq \overline{\alpha}$. Then $\overline{\gamma} \preceq \overline{\alpha}$.*

Let us fix an approximation $\overline{\alpha}$ of α. We define the notions of $\overline{\alpha}$-regular finite part, $\overline{\alpha}$-rank and $\overline{\alpha}$-forcing by means of transfinite recursion over α.

(i) Let first $\alpha = 0$. Then $\overline{\alpha} = \langle 0 \rangle$. 0-regular are those finite parts satisfying the condition:

If $z \in 2\mathbf{N} + 1$, $z \in \text{dom}(\tau)$ and $\tau(z) \neq \perp$, then $\tau(z) \in B_0$.

If $\text{dom}(\tau) = [0, 2q+1]$ we set the 0-rank $|\tau|_0$ of τ to be $q + 1$.

We will use the notation $\mathcal{R}_0$ for the set of all 0-regular finite parts.

For arbitrary finite part ρ we define:

$$\rho \Vdash_0 F_i(x) \iff \exists v(\langle x, v \rangle \in W_i \ \&\ D_v \subseteq \langle \tau \rangle),$$
$$\rho \Vdash_0 \neg F_i(x) \iff (\forall \tau \in \mathcal{R}_0)(\tau \supseteq \rho \Longrightarrow \tau \not\Vdash_0 F_i(x)).$$

Now suppose that for all $\beta < \alpha$ the $\overline{\beta}$-regularity, $\overline{\beta}$-rank and $\overline{\beta}$-forcing are defined. We will also assume that for all $\beta < \alpha$ the function $\overline{\beta}$-rank denoted by $\lambda\tau.|\tau|_{\overline{\beta}}$ has the property:

If τ and ρ are two $\overline{\beta}$-regular finite parts such that $\tau \subseteq \rho$, then $|\tau|_{\overline{\beta}} \leq |\rho|_{\overline{\beta}}$. In particular $|\tau|_{\overline{\beta}} = |\rho|_{\overline{\beta}} \iff \tau = \rho$.

(ii) Let now $\alpha = \beta + 1$. Let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. Denote the set of all $\overline{\beta}$-regular finite parts by $\mathcal{R}_{\overline{\beta}}$. Let also

$$X^{\overline{\beta}}_{\langle i,j \rangle} = \{\rho \in \mathcal{R}_{\overline{\beta}} \mid \rho \Vdash_{\overline{\beta}} F_i(j)\},$$
$$S^{\overline{\beta}}_j = \mathcal{R}_{\overline{\beta}} \cap \Gamma_j(\mathcal{P}_\beta),$$

where Γ_j is the j-th enumeration operator.

If ρ is an arbitrary finite part and X is a set of $\overline{\beta}$-regular finite parts we define the function $\mu_{\overline{\beta}}(\rho, X)$ by:

$$\mu_{\overline{\beta}}(\rho, X) = \begin{cases} \mu\tau[\tau \supseteq \rho \ \&\ \tau \in X], & \text{if there is such } \tau \qquad \text{(a)} \\ \mu\tau[\tau \supseteq \rho \ \&\ \tau \in \mathcal{R}_{\overline{\beta}}], & \text{if (a) is not satisfiable} \qquad \text{(b)} \\ \neg !, & \text{if (a) and (b) are not satisfiable} \qquad \text{(c)} \end{cases}$$

Definition 3.3. Let τ be a finite part and let $m \geq 0$. We say that ρ is $\overline{\beta}$-regular m-omitting extension of τ, iff ρ is $\overline{\beta}$-regular extension of τ, defined in $[0, q-1]$ and there are natural numbers $q_0 < q_1 < \cdots < q_m < q_{m+1} = q$ such that

a) $\rho \restriction q_0 = \tau$

b) for all $p \leq m$, it is true that $\rho \restriction q_{p+1} = \mu_{\overline{\beta}}\left(\rho \restriction (q_p + 1), X^{\overline{\beta}}_{\langle p, q_p \rangle}\right)$.

It is clear that if ρ is $\overline{\beta}$-regular m-omitting extension of τ, then $q_0, q_1, \ldots, q_{m+1}$ are unique. Even more: if ρ_1 and ρ_2 are two $\overline{\beta}$-regular m-omitting extensions of τ and $\rho_1 \subseteq \rho_2$ then $\rho_1 = \rho_2$. In other case the function $\mu_{\overline{\beta}}$ is not single valued.

Now we are ready to define the notion of $\overline{\alpha}$-regular finite part:

Let τ be a finite part defined in $[0, q-1]$ and let $r \geq 0$. We say that τ is an $\overline{\alpha}$-regular finite part with $\overline{\alpha}$-rank $r+1$ iff there are natural numbers

$$0 < n_0 < l_0 < b_0 < n_1 < l_1 < b_1 < \cdots < n_r < l_r < b_r < n_{r+1} = q,$$

such that for all $0 \leq j \leq r$ the following assertions hold:

(1) $\tau \restriction n_0$ *is a $\overline{\beta}$-regular finite part of $\overline{\beta}$-rank 1;*

(2) $\tau \restriction l_j = \mu_{\overline{\beta}}\left(\tau \restriction (n_j + 1), S^{\overline{\beta}}_j\right)$;

(3) $\tau \restriction b_j$ *is $\overline{\beta}$-regular j-omitting extension of $\tau \restriction l_j$;*

(4) $\tau(b_j) \in B_\alpha$;

(5) $\tau \restriction n_{j+1}$ *is a $\overline{\beta}$-regular extension of $\tau \restriction (b_j + 1)$ of rank $|\tau \restriction b_j|_{\overline{\beta}} + 1$.*

Note that directly from the definition it follows that if τ is an $\overline{\alpha}$-regular finite part, then τ is also a $\overline{\beta}$-regular finite part.

The definition of $\overline{\alpha}$-forcing for an arbitrary finite part ρ is:

$$\begin{aligned} \rho \Vdash_{\overline{\alpha}} F_i(x) \iff \exists v(\langle v, x\rangle \in W_i \;\&\; (\forall u \in D_v) \Big(& (u = \langle i_u, x_u, 0\rangle \;\&\; \rho \Vdash_{\overline{\beta}} F_{i_u}(x_u)) \\ & \vee (u = \langle i_u, x_u, 1\rangle \;\&\; \rho \Vdash_{\overline{\beta}} \neg F_{i_u}(x_u))\Big) \end{aligned}$$

$$\rho \Vdash_{\overline{\beta}} \neg F_i(x) \iff (\forall \tau \in \mathcal{R}_{\overline{\alpha}})(\rho \subseteq \tau \Longrightarrow \tau \nVdash_{\overline{\alpha}} F_i(x))$$

(iii) Finally let $\alpha = \underline{\lim}\, \alpha(p)$. Let $\overline{\alpha} = \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha$ and let $p_0 = \mu p[\alpha(p) > \alpha_n]$. Let also for all p, $\overline{\alpha(p)}$ be the $\alpha(p)$-predecessor of $\overline{\alpha}$. Note that for $p \geq p_0$ according to Lemma 3.2

$$\overline{\alpha(p)} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \ldots, \alpha(p)\rangle.$$

We say that the finite part τ defined for $[0, q-1]$ is $\overline{\alpha}$-regular of $\overline{\alpha}$-rank $r+1$ if there are natural numbers

$$0 < n_0 < b_0 < n_1 < b_1 < \cdots < n_r < b_r < n_{r+1} = q,$$

such that $0 \leq j \leq r$, it is true that:

(1) $\tau \upharpoonright n_0$ *is a* $\langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$*-regular finite part of rank* 1;

(2) $\tau \upharpoonright b_j$ *is a* $\overline{\alpha(p_0 + 2j)}$*-regular finite part of rank* 1;

(3) $\tau(b_j) \in B_\alpha$;

(4) $\tau \upharpoonright n_{j+1}$ *is a* $\overline{\alpha(p_0 + 2j + 1)}$*-regular finite part of rank* 1.

Note that in this case, τ is a $\overline{\alpha(p_0 + 2r + 1)}$-regular finite part of respectivly rank 1.

For every finite part ρ and every $i, x \in \mathbf{N}$ we define:

$$\rho \Vdash_{\overline{\alpha}} F_i(x) \Longleftrightarrow \exists v \Big(\langle v, x \rangle \in W_i \,\&\, (\forall u \in D_v)(u = \langle p_u, i_u, x_u \rangle \,\&\, \rho \Vdash_{\overline{\alpha(p_u)}} F_{i_u}(x_u)) \Big),$$
$$\rho \Vdash_{\overline{\alpha}} \neg F_i(x) \Longleftrightarrow (\forall \tau \in \mathcal{R}_{\overline{\alpha}})(\rho \subseteq \tau \Rightarrow \tau \nVdash_{\overline{\alpha}} F_i(x)).$$

This concludes the definition. The next Lemma gives the correctness of the definition and the validity of the assumption for the $\overline{\beta}$-rank.

Lemma 3.4. *Let* $\alpha \leq \zeta$ *and let* τ *be* $\overline{\alpha}$*-regular finite part. Then the following statements are true:*

(a) Let $\alpha = \beta + 1$. *Let also* $n'_0, l'_0, b'_0, \dots n'_r,\ l'_r, b'_r, n'_{r+1}$ *and* $n''_0, l''_0,\ b''_0, \dots\ n''_p,$ l''_p, b''_p, n''_{p+1} *be two sequences of natural numbers satisfying (1)-(5) from* **(ii)**. *Then* $r = p$, $n'_{r+1} = n''_{r+1}$ *and for all* $0 \leq j \leq r$ *we have* $n'_j = n''_j$, $l'_j = l''_j$ *and* $b'_j = b''_j$.

(b) Let $\alpha = \lim \alpha(p)$ *and let* $n'_0, b'_0, \dots n'_r, b'_r, n'_{r+1}$ *and* $n''_0, b''_0, \dots n''_p, b''_p, n''_{p+1}$ *are two sequences of natural numbers satisfying (1)-(4) from* **(iii)**. *Then* $r = p$, $n'_{r+1} = n''_{r+1}$ *and for all* $0 \leq j \leq r$ *we have* $n'_j = n''_j$ *and* $b'_j = b''_j$.

(c) Let ρ *and* τ *be* $\overline{\alpha}$*-regular finite parts and let* $\tau \subseteq \rho$. *Then* $|\tau|_{\overline{\alpha}} \leq |\rho|_{\overline{\alpha}}$. *In particular* $|\tau|_{\overline{\alpha}} = |\rho|_{\overline{\alpha}} \iff \tau = \rho$.

Proof. (a) Let $\alpha = \beta + 1$ and let $n'_0, l'_0, b'_0, \dots, n'_r, l'_r, b'_r, n'_{r+1}$ and $n''_0, l''_0,\ b''_0, \dots,$ $n''_p,\ l''_p, b''_p, n''_{p+1}$ be two sequences of natural numbers satisfying (1)-(5) from **(ii)**. Without loss of generality we may assume that $\tau \upharpoonright n'_0 \subseteq \tau \upharpoonright n''_0$. Beside this, we have $|\tau \upharpoonright n'_0|_{\overline{\beta}} = |\tau \upharpoonright n''_0|_{\overline{\beta}} = 1$. Then considering the properties of $\overline{\beta}$-rank we obtain $\tau \upharpoonright n'_0 = \tau \upharpoonright n''_0$. Therefore $n'_0 = n''_0$. Let now the equality $n'_j = n''_j$ hold. Then $\tau \upharpoonright l'_j = \mu_{\overline{\beta}} \left(\tau \upharpoonright n'_j, S^{\overline{\beta}}_j \right) = \mu_{\overline{\beta}} \left(\tau \upharpoonright n''_j, S^{\overline{\beta}}_j \right) = \tau \upharpoonright l''_j$. Therefore $l'_j = l''_j$. Now considering the property of the j-omitting $\overline{\beta}$-regular extensions (mentioned after the definition) we obtain $\tau \upharpoonright b'_j = \tau \upharpoonright b''_j$ and therefore $b'_j = b''_j$. Now again without loss of generality we may consider $\tau \upharpoonright n'_{j+1} \subseteq \tau \upharpoonright n''_{j+1}$. But $|\tau \upharpoonright n'_{j+1}|_{\overline{\beta}} = |\tau \upharpoonright b'_j|_{\overline{\beta}} + 1 = |\tau \upharpoonright b''_j|_{\overline{\beta}} + 1 = |\tau \upharpoonright n''_{j+1}|_{\overline{\beta}}$. Therefore from the property of the $\overline{\beta}$-rank we obtain $n'_{j+1} = n''_{j+1}$. Now the statement $r = p$ is obvious.

(b) The proof is analogous to the previous one.

(c) Let τ and ρ be two $\overline{\alpha}$-regular finite parts and let $\tau \subseteq \rho$. From the proof of (a) we obtain that the sequence corresponding to τ and satisfying the definition of the $\overline{\alpha}$-regular finite parts is an initial part of the sequence corresponding to ρ.

Therefore $|\tau|_{\overline{\alpha}} \leq |\rho|_{\overline{\alpha}}$. If $\tau \subsetneq \rho$ then we have $|\tau|_{\overline{\alpha}} < |\rho|_{\overline{\alpha}}$, since in the contrary case we would obtain that the sequence of ρ is not monotone. □

From the definition of $\overline{\alpha}$-regular finite part and Lemma 3.4 we obtain

Corollary 3.1. *Let $\alpha = \beta + 1$, $\overline{\alpha}$ be an approximation of α and let $\overline{\beta}$ be β-predecessor of $\overline{\alpha}$. Then every $\overline{\alpha}$-regular finite part τ is $\overline{\beta}$-regular and $|\tau|_{\overline{\beta}} > |\tau|_{\overline{\alpha}}$.*

Lemma 3.5. *Let $1 \leq \alpha \leq \zeta$ and let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha \rangle$. Then every $\overline{\alpha}$-regular finite part is $\langle \alpha_0, \dots, \alpha_n \rangle$-regular and the $\langle \alpha_0, \dots, \alpha_n \rangle$-rank of τ is strictly greater than $|\tau|_{\overline{\alpha}}$.*

Proof. We will use transfinite induction over α. First let $\alpha = 1$. Then $\overline{\alpha} = \langle 0, 1 \rangle$ and now the statement follows from Corollary 3.1.

Let now $\alpha = \beta + 1$ and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. Then again (from Corollary 3.1) we obtain that τ is $\overline{\beta}$-regular finite part and $|\tau|_{\overline{\beta}} > |\tau|_{\overline{\alpha}}$. From Lemma 3.2 we know that $\overline{\beta}$ is of the form $\langle \alpha_0, \alpha_1, \dots, \alpha_n, \beta_{n+1}, \dots, \beta_{n+i} \rangle$, where $i \geq 0$. Then applying i times the induction hypothesis we obtain that τ is $\langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$-regular and the $\langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$-rank of τ is greater or equal to $|\tau|_{\overline{\beta}}$ and therefore strictly greater than $|\tau|_{\overline{\alpha}}$.

Finally let $\alpha = \lim \alpha(p)$. Let also $|\tau|_{\overline{\alpha}} = r + 1$ and let $p_0 = \mu p[\alpha(p_0) > \alpha_n]$. From the definition of $\overline{\alpha}$-regular finite part we obtain that τ is a $\langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha(p_0), \dots, \alpha(p_0 + 2r + 1) \rangle$-regular finite part of rank 1. From the induction hypothesis τ is a $\langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha(p_0), \dots, \alpha(p_0 + 2r) \rangle$-regular finite part of rank at least 2 and since τ is a $\langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha(p_0) \rangle$-regular finite part of rank at least $2r+2$, then τ is $\langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$-regular of rank at least $2r+3$ and therefore strictly greater than $r + 1$. □

Lemma 3.6. *Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an approximation of α. let also $\overline{\delta} \preceq \overline{\alpha}$. Then there is a natural number $k_{\overline{\alpha},\overline{\delta}}$, such that every $\overline{\alpha}$-regular finite part of rank greater or equal to $k_{\overline{\alpha},\overline{\delta}}$ is $\overline{\delta}$-regular.*

Proof. We will use transfinite induction over α. When $\alpha = 0$ the statement is trivial.

Now let $\alpha = \beta + 1$ and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. Let $\overline{\delta} \prec \overline{\alpha}$ (which is the interesting case). Then $\overline{\delta} \preceq \overline{\beta}$. According to the induction hypothesis there is a $k = k_{\overline{\beta},\overline{\delta}}$, such that every $\overline{\beta}$-regular finite part of rank greater or equal to k is $\overline{\delta}$-regular. Let us set $k_{\overline{\alpha},\overline{\delta}} = k$. Then according to Corollary 3.1 we obtain that k has the desired property.

Finally let $\alpha = \lim \alpha(p)$, $\overline{\alpha} = \langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha \rangle$ and $\overline{\delta} \prec \overline{\alpha}$. Let also $p_0 = \mu p[\alpha(p) > \alpha_n]$, let $p_1 \geq p_0$ be such that $\alpha(p_1) > \delta$ and let us denote the $\alpha(p)$-predecessor of $\overline{\alpha}$ with $\overline{\alpha(p)}$. Applying Lemma 3.2 we obtain $\overline{\delta} \preceq \overline{\alpha}(p_1)$. Then according to the induction hypothesis every $\overline{\alpha}(p_1)$-regular finite part with rank greater or equal to $k_{\overline{\alpha}(p_1),\overline{\delta}}$ is $\overline{\delta}$-regular. It follows from the proof of the previous

Lemma that there is a natural number r, such that every $\overline{\alpha}$-regular finite part of rank at least $r+1$ is $\overline{\alpha}(p_1)$-regular of rank greater or equal to $k_{\overline{\alpha}(p_1),\overline{\delta}}$. Let us set $k_{\overline{\alpha},\overline{\beta}} = r+1$ □

Corollary 3.2. *Let $\alpha \leq \zeta$, $\overline{\alpha}$ be an approximation of α and $\overline{\beta} \preceq \overline{\alpha}$. Let also τ be $\overline{\alpha}$-regular finite part of rank greater or equal to $k_{\overline{\alpha},\overline{\beta}} + s$. Then $|\tau|_{\overline{\beta}} > s$.*

Proof. From the definition of the $\overline{\alpha}$ regular finite parts we obtain that there are natural numbers $q_0 < q_1 < \cdots < q_s$ such that $\tau \restriction q_s = \tau$ and for all j the finite parts $\tau_j = \tau \restriction q_j$ are $\overline{\alpha}$-regular with $\overline{\alpha}$-rank at least $k_{\overline{\alpha},\overline{\beta}}$ and therefore $\overline{\beta}$-regular. But $\tau_0 \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau_s$ and therefore $|\tau_j|_{\overline{\beta}} < |\tau_{j+1}|_{\overline{\beta}}$. Finally $|\tau_0|_{\overline{\beta}} \geq 1$, which completes the proof. □

Lemma 3.7. *Let $\alpha = \lim \alpha(p)$. Let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha \rangle$ and $p_0 = \mu p[\alpha(p) > \alpha_n]$. Let also $p_1 \geq p_0$ and τ be a $\langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha(p_0), \alpha(p_0+1), \dots, \alpha(p_1) \rangle$-regular finite part of rank 1. Then for every $\overline{\beta} \prec \overline{\alpha}$, if τ is $\overline{\beta}$-regular then $\beta \leq \alpha(p_1)$.*

Proof. In order to obtain a contradiction assume that τ is a $\overline{\beta}$-regular finite part for some β such that $\overline{\beta} \prec \overline{\alpha}$ and $\alpha(p_1) < \beta < \alpha$. Then $\overline{\beta}$ is the β-predecessor of

$$\langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha(p_0), \alpha(p_0+1), \dots, \alpha(p_1+k) \rangle,$$

where $k \geq 1$. According to Lemma 3.2 $\overline{\beta}$ is of the form

$$\langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha(p_0), \dots, \alpha(p_1), \dots, \beta \rangle.$$

As the $\overline{\beta}$-rank of τ is at least 1 then from Lemma 3.5 we obtain that the $\langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha(p_0), \dots, \alpha(p_1) \rangle$-rank of τ is greater than 1 which is a contradiction. □

Let $\overline{\alpha}$ be an ordinal approximation and let τ be a finite part. We introduce the following notation:

$$Reg(\tau, \overline{\alpha}) = \{ \overline{\beta} \mid \overline{\beta} \preceq \overline{\alpha} \;\&\; \tau \text{ is } \overline{\beta}\text{-regular} \}$$

Then the following is true:

Lemma 3.8. *Let $\alpha \leq \zeta$, let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \dots, \alpha_n, \alpha \rangle$ be an approximation of α and let τ be an $\overline{\alpha}$-regular finite part. Then:*

a) if $\alpha = \delta + 1$ and $\overline{\delta}$ is the δ-predecessor of $\overline{\alpha}$ then

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \iff \overline{\beta} = \overline{\alpha} \vee \overline{\beta} \in Reg(\tau, \overline{\delta});$$

b) let $\alpha = \lim \alpha(p)$. Let also $p_0 = \mu p[\alpha(p) > \alpha_n]$ and for every $p \geq p_0$ let $\overline{\alpha(p)}$ be $\alpha(p)$-predecessor of $\overline{\alpha}$. Let also $p_1 \geq p_0$ and let τ be $\overline{\alpha(p_1)}$-regular of rank 1. Then

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \iff \overline{\beta} = \overline{\alpha} \vee \overline{\beta} \in Reg(\tau, \overline{\alpha(p_1)}).$$

Proof. The statement *a)* is obvious and the statement *b)* follows directly from the previous Lemma. □

Definition 3.4. We say that the sequence $A_0, \ldots, A_n, \ldots$ of sets of natural numbers is *e*-reducible to P iff there is a recursive function h such that for every n $A_n = \Gamma_{h(n)}(P)$. We say that the sequence is T-reducible to P iff there is a function χ recursive in P, such that for every n $\lambda x.\chi(n,x) = \chi_{A_n}$, where χ_{A_n} is the characteristic function of A_n.

From the definition of the enumeration jump, the e-reducibility and the T-reducibility of sequences to set we obtain the following Lemma.

Lemma 3.9. *Let P be a set such that the sequence $\{A_n\}$ is e-reducible to P. Then*

(1) *The sequence $\{A_n\}$ is uniformly T-reducible to P';*

(2) *If $R \leq_e P$ then the sequences $\{A_n \cap R\}$ and $\{C_n\}$ for which $C_n = \{\, x \mid \exists y(\langle y, x\rangle \in R \;\&\; y \in A_n\}$ are uniformly e-reducible to P.*

The full proof can be found in [2].

We introduce the following notations:

$$\begin{aligned} Z^{\overline{\alpha}}_{\langle i,j\rangle} &= \{\tau \in \mathcal{R}_{\overline{\alpha}} \mid \tau \Vdash_{\overline{\alpha}} \neg F_i(j)\} \\ O^{\overline{\alpha}}_{\tau,j} &= \{\rho \mid \rho \text{ is } \overline{\alpha}\text{-regular } j\text{-omitting extension of } \tau\} \end{aligned}$$

Proposition 3.1. *For every ordinal approximation $\overline{\alpha}$, where $\alpha \leq \zeta$ the following are true:*

(1) *$\mathcal{R}_{\overline{\alpha}} \leq_e \mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$.*

(2) *The function $\lambda\tau.|\tau|_{\overline{\alpha}}$ is partially recursive in $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$;*

(3) *The sequences $\{S^{\overline{\alpha}}_j\}$ and $\{X^{\overline{\alpha}}_j\}$ are e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$;*

(4) *The sequence $\{Z^{\overline{\alpha}}_j\}$ is T-reducible to $\mathcal{P}'_\alpha$ uniformly in $\overline{\alpha}$;*

(5) *the functions $\lambda\tau, j.\mu_{\overline{\alpha}}(\tau, X^{\overline{\alpha}}_j)$ and $\lambda\tau, j.\mu_{\overline{\alpha}}(\tau, S^{\overline{\alpha}}_j)$ are partially recursive in $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$;*

(6) *The sequence $\{O^{\overline{\alpha}}_{\tau,j}\}$ is e-reducible to $\mathcal{P}'_\alpha$ uniformly $\overline{\alpha}$.*

Before proving the proposition let us note some properties of the sets $\mathcal{P}_\alpha$.

Lemma 3.10. (a) *If $\beta \leq \alpha \leq \zeta$ then $\mathcal{P}_\beta \leq_e \mathcal{P}_\alpha$ uniformly in α and β.*

(b) *If $\beta \leq \alpha \leq \zeta$ then $B_\beta \leq_e \mathcal{P}_\alpha$ uniformly in α and β;*

(c) *The sets $\mathcal{P}_{<\alpha}$ are total.*

Proof. (a) We must find a recursive function g, such that if $\beta \leq \alpha \leq \zeta$ then $\mathcal{P}_\beta = \Gamma_{g(\alpha,\beta)}(\mathcal{P}_\alpha)$. We will define g by recursion over the ordinals $\alpha \leq \zeta$. If $\alpha = 0$ then $g(0,0) = i_0$, where i_0 is a fixed index for the enumeration operator identity. If $\alpha = \beta$ then again $g(\alpha, \beta) = i_0$. Now let $\beta < \alpha$.

First consider $\alpha = \delta + 1$. Then $\mathcal{P}_\beta \leq_e \mathcal{P}_\delta$ and therefore $\mathcal{P}_\beta = \Gamma_{g(\delta,\beta)}(\mathcal{P}_\delta)$. But $\mathcal{P}_\delta = \Gamma_{j_0}(\Gamma_{p_0}(\mathcal{P}_\alpha))$, where j_0 is a fixed index for which $A = \Gamma_{j_0}(A')$ and p_0 is such that $A = \Gamma_{p_0}(A \oplus C)$ (j_0 and p_0 exist and do not depend on A and C). Then

$$g(\alpha,\beta) = \mathfrak{c}(g(\delta,\beta), \mathfrak{c}(j_0, p_0)).$$

For the definition of $\mathfrak{c}$ see Section 2.

Finally let $\alpha = \lim \alpha(p)$. Then there is a recursive function pr not depending on α, such that $\mathcal{P}_{\alpha(i)} = \Gamma_{pr(i)}(\mathcal{P}_{<\alpha})$. The function $m(\alpha,\beta) = \mu p[\alpha(p) \geq \beta]$, defined for the limit ordinals $\alpha \leq \zeta$ and all ordinals $\beta < \alpha$, is partially recursive. Then $\mathcal{P}_\beta \leq_e \mathcal{P}_{m(\alpha,\beta)}$ and $\mathcal{P}_{m(\alpha,\beta)} = \Gamma_{pr(m(\alpha,\beta))}(\mathcal{P}_{<\alpha})$. We set

$$g(\alpha,\beta) = \mathfrak{c}\,(g(m(\alpha,\beta),\beta), \mathfrak{c}\,(pr(m(\alpha,\beta)), p_0)).$$

(b) Follows directly from (a).

(c) Let $\alpha = \lim \alpha(p)$. We must show that $\mathbf{N}\backslash\mathcal{P}_{<\alpha} \leq_e \mathcal{P}_{<\alpha}$. Recall that $\mathcal{P}_{<\alpha} = \{\langle p, x\rangle \mid x \in \mathcal{P}_{\alpha(p)}\}$. Therefore $x \in \mathbf{N}\backslash\mathcal{P}_{<\alpha} \iff x \notin \mathcal{P}_{<\alpha} \iff x = \langle p, y\rangle \;\&\; y \notin \mathcal{P}_{\alpha(p)}$. Now according to the definition of the enumeration jump we obtain that for arbitrary set C and every z

$$z \notin C \iff 2\langle z, i_0\rangle + 1 \in C',$$

where i_0 is a fixed index for the enumeration operator identity. Now from the proof of (a) we obtain that the sequence $\mathcal{P}'_{\alpha(p)}$ is e-reducible to $\mathcal{P}_{<\alpha}$ uniformly in $\alpha(p)$ and therefore the condition $x \in \mathbf{N}\backslash\mathcal{P}_{<\alpha}$ is e-reducible to $\mathcal{P}_{<\alpha}$. □

Proof of Lemma 3.1. Transfinite induction over α. In the case $\alpha = 0$ the statements are clear. Now let the statements be true for every $\delta < \alpha$. First we will prove (1).

(1) First consider $\alpha = \beta + 1$ and let τ be an arbitrary finite part. Then we set the number n_0 to be $n_0 = \mu q[\tau \restriction q \in \mathcal{R}_{\overline{\beta}}]$. Finding n_0 or proving that such number does not exist is recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$, since according to the induction hypothesis $\mathcal{R}_{\overline{\beta}} \leq_e \mathcal{P}_\beta$ uniformly in $\overline{\beta}$. If there is no such n_0 then $\tau \notin \mathcal{R}_{\overline{\beta}}$. Let n_j be defined for some $j \geq 0$. Then, if $\mu_{\overline{\beta}}\left(\tau \restriction n_j, S_j^{\overline{\beta}}\right)$ is defined and $\mu_{\overline{\beta}}\left(\tau \restriction n_j, S_j^{\overline{\beta}}\right) \subseteq \tau$, we set $l_j = \mathrm{lh}\left(\mu_{\overline{\beta}}\left(\tau \restriction n_j, S_j^{\overline{\beta}}\right)\right)$. Since the function $\mu_{\overline{\beta}}$ is partially recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$, defining l_j is r.e. in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$. If we have defined l_j then we set

$$b_j = \mu q[\,q > l_j \;\&\; \tau \restriction q \in O^{\overline{\beta}}_{\langle \tau \restriction l_j, j\rangle}]$$

We know from the induction hypothesis that the sets $O^{\overline{\beta}}_{\langle \rho, j\rangle}$ are e-reducible to $\mathcal{P}'_\beta$ (which is a total set) uniformly in $\overline{\beta}$ and $\langle \rho, j\rangle$, and therefore setting b_j is again r.e. in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$. Finally if there is a q, such that $\tau \restriction q \in \mathcal{R}_{\overline{\beta}}$, we set

$$n_{j+1} = \mu q[q > b_j + 1 \ \& \ \tau \restriction q \in \mathcal{R}_{\overline{\beta}}]$$

Knowing b_j, defining n_{j+1} is recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$, and therefore is r.e. in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$. Then $\tau \in \mathcal{R}_{\overline{\beta}}$ iff there is n_{r+1}, which is obtained following the construction above, such that $\tau \restriction n_{r+1} = \tau$ and for every $0 \leq j \leq r$ it is true that $\tau(b_j) \in B_\alpha$. The first condition is r.e. in the total set $\mathcal{P}'_\beta$. The second one is e-reducible to B_α. The two of them are uniform in $\overline{\alpha}$. Therefore $\mathcal{R}_{\overline{\alpha}} \leq_e \mathcal{P}'_\beta \oplus B_\alpha$.

Now consider $\alpha = \lim \alpha(p)$. Let τ be an arbitrary finite part. According to Lemma 3.10 we obtain that the sequence $\{\mathcal{P}_{\alpha(p)}\}$ is e-reducible to $\mathcal{P}_{<\alpha}$ uniformly in $\overline{\alpha}$. Since the sets $\mathcal{R}_{\overline{\alpha(p)}}$ are e-reducible to $\mathcal{P}_{\alpha(p)}$ uniformly in $\overline{\alpha(p)}$, we obtain that the sequence $\{\mathcal{R}_{\overline{\alpha(p)}}\}$ is e-reducible to $\mathcal{P}_{<\alpha}$ uniformly in $\overline{\alpha}$. Analogously to the case $\alpha = \beta + 1$, we can find r.e. in $\mathcal{P}_{<\alpha}$ and uniformly in $\overline{\alpha}$ a sequence of numbers $n_0, b_0, n_1, b_1, \ldots$ satisfying the conditions of the definition of the $\overline{\alpha}$-regularity of τ. If for some of the numbers n_{r+1} is true that $n_{r+1} = \mathrm{lh}(\tau)$ and for every $0 \leq j \leq r$ $\tau(b_j) \in B_\alpha$ then $\tau \in \mathcal{P}_\alpha$. These conditions are e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$.

(2) Follows directly from the proof of (1).

(3) The sequence $\{S_j^{\overline{\alpha}}\}$ is e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$ as $S_j^{\overline{\alpha}} = \mathcal{R}_{\overline{\alpha}} \cap \Gamma_j(\mathcal{P}_\alpha)$ (Lemma 3.9). In order to prove the statement for $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ let us first assume that $\alpha = \beta + 1$. According to the definition $X^{\overline{\alpha}}_{\langle i,j\rangle} = \{\tau \in \mathcal{R}_{\overline{\alpha}} \mid \tau \Vdash_{\overline{\alpha}} F_i(j)\}$. Also

$$\tau \Vdash_{\overline{\alpha}} F_i(j) \iff \exists v\,(\langle j, v\rangle \in W_i \ \&$$

$$(\forall u \in D_v)((u = \langle 0, i_u, x_u\rangle \ \& \ \tau \Vdash_{\overline{\beta}} F_{i_u}(x_u)) \vee (u = \langle 1, i_u, x_u\rangle \ \& \ \tau \Vdash_{\overline{\beta}} \neg F_{i_u}(x_u))$$

According to the induction hypothesis the conditions $\tau \Vdash_{\overline{\beta}} F_{i_u}(x_u))$ and $\tau \Vdash_{\overline{\beta}} \neg F_{i_u}(x_u))$ are recursive in $\mathcal{P}'_\beta$ uniformly in i_u, x_u and $\overline{\beta}$ (the sequences $\{X_k^{\overline{\beta}}\}$ and $\{Z_k^{\overline{\beta}}\}$ are T-reducible to $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$). Therefore the condition $\tau \Vdash_{\overline{\alpha}} F_i(j)$ is e-reducible to$\mathcal{P}'_\beta$ uniformly in i, j and $\overline{\beta}$. Therefore the sequence $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ is e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$.

Now let $\alpha = \lim \alpha(p)$. Then

$$\tau \Vdash_{\overline{\alpha}} F_i(j) \iff \exists v(\langle j, v\rangle \in W_i \ \& \ (\forall u \in D_v)(u = \langle p_u, i_u, x_u\rangle \ \& \ \tau \Vdash_{\overline{\alpha}(p)} F_{i_u}(x_u)))$$

But the sequence $\{\mathcal{P}_{\alpha(p)}\}$ is e-reducible to $\mathcal{P}_{<\alpha}$ uniformly in α. The sets $X^{\overline{\alpha}(p)}_{\langle i,j\rangle}$ are e-reducible to $\mathcal{P}_{\alpha(p)}$ uniformly in i, j and $\overline{\alpha}(p)$. Therefore the sequence $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ is e-reducible to $\mathcal{P}_{<\alpha}$ uniformly in $\overline{\alpha}$. As $\mathcal{P}_{<\alpha}$ is a total set the sequence $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ is r.e. in $\mathcal{P}_{<\alpha}$ uniformly in $\overline{\alpha}$. Then the condition $\tau \Vdash_{\overline{\alpha}(p_u)} F_{i_u}(x_u)$, if $\tau \in X^{\overline{\alpha}(p_u)}_{\langle i_u, x_u\rangle}$ is r.e. $\mathcal{P}_{<\alpha}$ uniformly in $\overline{\alpha}$. Finally we obtain that the sequence $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ is e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$.

(4) Since the sequence $\{X^{\overline{\alpha}}_{\langle i,j\rangle}\}$ is e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$ then the condition, for given τ it is true that $(\exists \rho \in X_i^{\overline{\alpha}})(\rho \supseteq \tau)$, is r.e. in $\mathcal{P}_\alpha$ uniformly

in i and $\overline{\alpha}$. Then the question, if for given τ is true that $(\forall \rho \supseteq \tau)(\rho \notin X_i^{\overline{\alpha}})$, i.e., if $\tau \in Z_i^{\overline{\alpha}}$, is r.e. in $\mathcal{P}'_\alpha$ uniformly in i and $\overline{\alpha}$. Therefore the sequence $\{Z_i^{\overline{\alpha}}\}$ is T-reducible to $\mathcal{P}'_\alpha$ uniformly in $\overline{\alpha}$.

(5) Follows directly from the definition of the function $\mu_{\overline{\alpha}}$ and the proof of (4).

(6) The reasoning is analogous to the proof of (1) and uses the fact that the function $\lambda\tau, i.\mu_{\overline{\alpha}}(\tau, X_i^{\overline{\alpha}})$ is partially recursive in $\mathcal{P}'_\alpha$ uniformly in $\overline{\alpha}$. □

Definition 3.5. Let τ be $\overline{\alpha}$-regular finite part with rank $r+1$. We define $B^{\tau}_{\overline{\alpha}}$ by:

a) if $\alpha = 0$, then $B^{\tau}_{\overline{\alpha}} = \{x \mid x \in \text{dom}\,(\tau)\ \&\ x \in 2\mathbf{N}+1\}$

b) if $\alpha = \beta + 1$ and $n_0, l_0, b_0, \ldots, n_r, l_r, b_r, n_{r+1}$ are the numbers from the definition of the regular parts, then $B^{\tau}_{\overline{\alpha}} = \{b_0, b_1, \ldots, b_r\}$

c) if $\alpha = \lim \alpha(p)$ and $n_0, b_0, \ldots, n_r, b_r, n_{r+1}$ are the numbers from the definition of the regular parts, then $B^{\tau}_{\overline{\alpha}} = \{b_0, b_1, \ldots, b_r\}$.

Definition 3.6. Let $\overline{\zeta}$ be an approximation of ζ. We say that the partial function f from $\mathbf{N}$ in $\mathbf{N}$ is a regular enumeration respecting $\overline{\zeta}$ iff:

(1) for every finite $\rho \subseteq f$ there is a $\overline{\zeta}$-regular finite part $\tau \supseteq \rho$ such that $\tau \subseteq f$;

(2) if $\overline{\alpha} \preceq \overline{\zeta}$ and $z \in B_\alpha$ then there is an $\overline{\alpha}$-regular $\tau \subseteq f$ such that $z \in \tau(B^{\tau}_{\overline{\alpha}})$.

It is clear from the definition, that if f is a regular enumeration, then f has $\overline{\zeta}$-regular subparts with arbitrary large rank. Then if $\overline{\alpha} \preceq \overline{\zeta}$ and $\rho \subseteq f$ there is an $\overline{\alpha}$-regular finite part $\tau \subseteq f$ such that $\rho \subseteq \tau$. In particular there are $\overline{\alpha}$-regular finite subparts of f of arbitrary rank.

If f is regular and $\overline{\alpha} \preceq \overline{\zeta}$ then with $B^{f}_{\overline{\alpha}}$ we will denote the set

$$B^{f}_{\overline{\alpha}} = \{b \mid (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\alpha}}\ \&\ b \in B^{\tau}_{\overline{\alpha}})\}.$$

It is clear that $f(B^{f}_{\overline{\alpha}}) = B_\alpha$.

Proposition 3.2. *Let f be a regular enumeration. Then:*

(1) $B_0 \leq_e f$;

(2) *if* $\alpha = \beta + 1 \leq \zeta$, *then* $B_\alpha \leq_e f^+ \oplus \mathcal{P}'_\beta$ *uniformly in* α;

(3) *if* $\alpha \leq \zeta$ *is a limit ordinal, then* $B_\alpha \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$ *uniformly in* α;

(4) $\mathcal{P}_\alpha \leq f^{(\alpha)}$ *uniformly in* α.

Proof. Let f be a regular enumeration. It is clear that $B^f_0 = 2\mathbf{N}+1$. It follows from the regularity that $B_0 = f(B^f_0)$. Therefore $B_0 \leq_e f$.

We will prove (2) and (3) using transfinite induction over α.

Let first $\alpha = \beta + 1$. Let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta}$, and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. Since f is a regular enumeration, then for every finite part $\rho \subseteq f$ there is an $\overline{\alpha}$-regular finite part $\tau \subseteq f$, such that $\rho \subseteq \tau$. Therefore there is a sequence of natural numbers

$$0 < n_0 < l_0 < b_0 < \cdots < n_r < l_r < b_r < \ldots,$$

satisfying the conditions from the definition of the $\overline{\alpha}$-regular finite parts, and also satisfying that $\tau_r = f \upharpoonright n_{r+1}$ is an $\overline{\alpha}$-regular finite part with $|\tau_r|_{\overline{\alpha}} = r+1$ for all $r \geq 0$. Therefore $B^f_{\overline{\alpha}} = \{b_0, b_1, \dots\}$. We will prove that there is a recursive in $f^+ \oplus \mathcal{P}'_\beta$, uniform in $\overline{\beta}$ procedure, which draws out the numbers $n_0, l_0, b_0, \dots$.

We know from the definition, that $\tau_0 = f \upharpoonright n_0$ is an $\overline{\alpha}$-regular finite part with rank $|\tau_0|_{\overline{\alpha}} = 1$. According to Proposition 3.1 the set $\mathcal{R}_{\overline{\beta}}$ is recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$. Using the oracle f^+ we may obtain successively all the finite parts $f \upharpoonright q$ for $q = 0, 1, \cdots$. Lemma 3.4 guarantees that τ_0 is the first from the so obtained finite parts which is in $\mathcal{R}_\beta$. Thus we obtain $n_0 = \mathrm{lh}(\tau_0)$.

Now let $r \geq -1$ and let the numbers $n_0, l_0, b_0, \dots, n_r, l_r, b_r, n_{r+1}$ have been obtained. As $S^{\overline{\beta}}_j$ is recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$, using the oracle $\mathcal{P}'_\beta$ we may obtain $f \upharpoonright l_{r+1} = \mu_{\overline{\beta}}\left(f \upharpoonright (n_{r+1}+1), S^{\overline{\beta}}_j\right)$. Thus we get $l_{r+1} = \mathrm{lh}(f \upharpoonright l_{r+1})$. We know that $f \upharpoonright b_{r+1}$ is a $\overline{\beta}$-regular, $r+1$-omitting extension of $f \upharpoonright l_{r+1}$. Therefore there are numbers $l_{r+1} = q_0 < q_1 < \cdots < q_{r+1} < q_{r+2} = b_{r+1}$ such that for every $p \leq r+1$, it is true that:

$$f \upharpoonright q_{p+1} = \mu_{\overline{\beta}}\left(f \upharpoonright (q_p+1), X^{\overline{\beta}}_{\langle p, q_p\rangle}\right).$$

Therefore, since the sets $X^{\overline{\beta}}_j$ are recursive in $\mathcal{P}'_\beta$ uniformly in $\overline{\beta}$, using successively the oracles f^+ and $\mathcal{P}'_\beta$ we may generate the finite parts $f \upharpoonright (q_p+1)$ for $p = 0, 1, \dots r+2$. At the end of this procedure we obtain the number b_{r+1}. In order to obtain n_{r+2} we generate using the oracle f^+ the finite parts $f \upharpoonright (b_{r+1}+1+q)$ for $q = 0, 1, \dots$. Then $n_{r+2} = \mathrm{lh}(f \upharpoonright n_{r+2})$, where $f \upharpoonright n_{r+2}$ is the first of the generated parts which is in $\mathcal{R}_{\overline{\beta}}$.

Thus we obtain that the set $B^f_\alpha = \{b_0, b_1, \dots\}$ is recursive in $f^+ \oplus \mathcal{P}'_\beta$ and therefore $B_\alpha = f(B^f_\alpha) \leq_e f^+ \oplus \mathcal{P}'_\beta$.

Now let $\alpha = \lim \alpha(p)$. It is clear, that the sequence $\{\mathcal{P}_{\alpha(p)}\}$ is uniformly e-reducible to $\mathcal{P}_{<\alpha}$. Let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta}$ and let $\overline{\alpha(p)}$ be the $\alpha(p)$-predecessor of $\overline{\alpha}$. Since f is a regular enumeration, we can assume that f is the union of $\overline{\alpha}$-regular finite parts. Therefore there are numbers

$$0 < n_0 < b_0 < n_1 < b_1 < \cdots < n_r < b_r < \dots$$

satisfying the conditions of the definition. Since for every p the sets $\mathcal{R}_{\overline{\alpha(p)}}$ are uniformly e-reducible to $\mathcal{P}'_{\alpha(p)}$, they are also uniformly e-reducible to $\mathcal{P}_{<\alpha}$. Hence applying the procedure from above we can get the numbers $n_0, b_0, \dots, n_r, b_r, \dots$ recursively in $f^+ \oplus \mathcal{P}_{<\alpha}$. Therefore $B_\alpha = f(B^f_\alpha) \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$.

Thus in both cases the sets B^f_α are r.e. in $f^+ \oplus \mathcal{P}'_\beta$ and $f^+ \oplus \mathcal{P}_{<\alpha}$, and besides this the procedures are uniform over β and α. Therefore the reducibilities in points (2) and (3) of the theorem are uniform over α.

We will prove statement (4) with transfinite induction over α.

In the case $\alpha = 0$ the statement is (1). Now let $\alpha = \beta+1$. Then $\mathcal{P}_\alpha = \mathcal{P}'_\beta \oplus B_\alpha$. According to the induction hypothesis $\mathcal{P}_\beta \leq_e f^{(\beta)}$ uniformly in β and therefore

$\mathcal{P}'_\beta \leq_e f^{(\alpha)}$ uniformly in α. Beside this $B_\alpha \leq_e f^+ \oplus \mathcal{P}'_\beta$ uniformly in $\overline{\alpha}$ and therefore $B_\alpha \leq_e f^{(\alpha)}$ uniformly in α. Therefore $\mathcal{P}_\alpha \leq_e f^{(\alpha)}$ uniformly in α.

Finally let $\alpha = \lim \alpha(p)$. Then $\mathcal{P}_\alpha = \mathcal{P}_{<\alpha} \oplus B_\alpha$. According to the induction hypothesis $\mathcal{P}_{\alpha(p)} \leq_e f^{(\alpha(p))}$ uniformly in $\overline{\alpha(p)}$. Therefore $\mathcal{P}_{\alpha(p)} \leq_e f^{(\alpha)}$ uniformly in $\alpha(p)$ and therefore $\mathcal{P}_{<\alpha} \leq_e f^{(\alpha)}$ uniformly in α. Beside this $B_\alpha \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$ and therefore $\mathcal{P}_\alpha \leq_e f^{(\alpha)}$ uniformly in α. □

Corollary 3.3. *Let f be a regular enumeration. Then $B_\alpha \leq_e f^{(\alpha)}$.*

Proof. From (5) of the proposition $\mathcal{P}_\alpha \leq f^\alpha$. But $B_\alpha \leq \mathcal{P}_\alpha$ which proves the corollary. □

Definition 3.7. Let f be a partial function from $\mathbf{N}$ to $\mathbf{N}$, let α be a recursive ordinal and let $i, x \in \mathbf{N}$. We define the relation $\models_\alpha$ by:

a) $\alpha = 0$
$f \models_0 F_i(x) \Leftrightarrow \exists v(\langle v, x\rangle \in W_i \;\&\; D_v \subseteq \langle f \rangle)$;

b) $\alpha = \beta + 1$
$$f \models_\alpha F_i(x) \Leftrightarrow \exists v(\langle v, x\rangle \in W_i \;\&\; (\forall u \in D_v)((u = \langle i_u, x_u, 0\rangle \;\&\; f \models_\beta F_{i_u}(x_u)) \vee (u = \langle i_u, x_u, 1\rangle \& f \models_\beta \neg F_{i_u}(x_u))));$$

c) $\alpha = \lim \alpha(p)$
$$f \models_\alpha F_i(x) \Leftrightarrow \exists v(\langle v, x\rangle \in W_i \& (\forall u \in D_v)(u = \langle p_u, i_u, x_u\rangle \& f \models_{\alpha(p_u)} F_{i_u}(x_u))).$$

d) for all other cases
$f \models_\alpha \neg F_i(x) \Leftrightarrow f \not\models_\alpha F_i(x)$.

The following Lemma is true:

Lemma 3.11. *There is a partial recursive function h such that for every recursive ordinal α and every enumeration operator Γ_i, it is true that*

$$x \in \Gamma_i(f^{(\alpha)}) \Longleftrightarrow f \models_\alpha F_{h(\alpha,i)}(x)$$

Before proving the Lemma let us note that for arbitrary set C if $\alpha = \beta + 1$ then

$$C^{(\alpha)} \equiv_e \{u \,|\, (u = \langle 0, i_u, x_u\rangle \,\& x_u \in \Gamma_{i_u}(C^{(\beta)})) \vee (u = \langle 1, i_u, x_u\rangle \,\& x_u \notin \Gamma_{i_u}(C^{(\beta)}))\},$$

and if $\alpha = \lim \alpha(p)$ then

$$C^{(\alpha)} \equiv_e \{u \mid u = \langle p_u, i_u, x_u\rangle \;\&\; x_u \in \Gamma_{i_u}(C^{(\alpha(p_u))})\}$$

uniformly in α.

Proof of Lemma 3.11. We will show that there is a sequence of recursive functions $\{\lambda j.h_\alpha(j)\}_{\alpha \leq \zeta}$ uniform in α such that for every $\alpha \leq \zeta$ and every i the statement

$$x \in \Gamma_i(f^{(\alpha)}) \Longleftrightarrow f \models_\alpha F_{h_\alpha(i)}(x)$$

holds. We will use transfinite induction over $\alpha \leq \zeta$. First let $\alpha = 0$. We set $h_0(i) = i$. It is clear from the definition of $\models_0$ that h_0 has the desired property. Now let $\alpha = \beta + 1$. Then

$$x \in \Gamma_i(f^{(\alpha)})$$

$$\Updownarrow$$

$$\exists v(\langle x, v\rangle \in W_i \;\&\; D_v \subseteq f^{(\alpha)})$$

$$\Updownarrow$$

$$\exists v(\langle x, v\rangle \in W_i \;\&\; (\forall u \in D_v)((u = \langle 0, i_u, x_u\rangle \;\&\; x_u \in \Gamma_{i_u}(f^{(\beta)}))\vee$$
$$(u = \langle 1, i_u, x_u\rangle \;\&\; x_u \notin \Gamma_{i_u}(f^{(\beta)}))).$$

Then from h_β we obtain

$$x \in \Gamma_i(f^{(\alpha)})$$

$$\Updownarrow$$

$$\exists v(\langle x, v\rangle \in W_i \;\&\; (\forall u \in D_v)((u = \langle 0, i_u, x_u\rangle \;\& f \models_\beta F_{h_\beta(i_u)}(x_u))\vee$$
$$(u = \langle 1, i_u, x_u\rangle \;\& f \not\models_\beta F_{h_\beta(i_u)}(x_u)))).$$

Consider the set W such that $\langle x, v\rangle \in W$ iff there exists v' such that $\langle x, v'\rangle \in W_i$ and

$$\forall\langle t, i, x\rangle(\langle t, h_\beta(i), x\rangle \in D_v \iff \langle t, i, x\rangle \in D_{v'})$$

Since the function h_β is recursive uniformly in β, then we can obtain recursively and uniformly in β the finite sets D_v from the finite sets $D_{v'}$. Therefore the set W is r.e. with Gödel index i_0. Thus we obtain $x \in \Gamma_{f^{(\alpha)}} \iff f \models_{i_0} (x)$. Beside this, W is obtained uniformly from the index i of the r.e. set W_i and the function h_β. Then i_0 is also obtained uniformly from i and h_β. We set $h_\alpha(i) = i_0$.

Finally let $\alpha = \lim \alpha(p)$. Then $x \in \Gamma_i(f^{(\alpha)}) \iff \exists v(\langle x, v\rangle \in W_i \;\&\; (\forall u \in D_v)(u = \langle p_u, i_u, x_u\rangle \;\& x_u \in \Gamma_{i_u}(f^{(\alpha(p_u))})))$. Then, according to the induction hypothesis $x \in \Gamma_i(f^{(\alpha)}) \iff \exists v(\langle x, v\rangle \in W_i \;\&\; (\forall u \in D_v)(u = \langle p_u, x_u, i_u\rangle \;\&$ $f \models_{\alpha(p_u)} F_{h_{\alpha(p_u)}(i_u)}(x_u)$. Let us consider the set W, for which $\langle x, v\rangle \in W$ iff there is a v' such that $\langle x, v'\rangle \in W_i$ and

$$\forall\langle p, i, x\rangle(\langle p, h_{\alpha(p)}(i), x\rangle \in D_v \iff \langle p, i, x\rangle \in D_{v'}).$$

Then, exactly as above (as the sequence of recursive functions $\{h_{\alpha(p)}\}$ is uniform in $\alpha(p)$), the finite sets D_v are obtained recursively from the finite sets $D_{v'}$, uniformly in $\{\alpha(p)\}$ and therefore uniformly in α. Then the set W is r.e. with index j_0, which is obtained uniformly from the index i and α. It is clear that $x \in \Gamma_i(f^{(\alpha)}) \iff f \models_\alpha F_{j_0}(x)$. We set $h_\alpha(i)$ to be $h_\alpha(i) = j_0$.

In both cases $h_\alpha(i)$ is uniformly obtained in i and α. □

Corollary 3.4. *Let f be a partial function from $\mathbf{N}$ to $\mathbf{N}$ and let α be a recursive ordinal. Then $A \leq_e f^{(\alpha)}$ iff there is an i such that for every x the condition $x \in A \iff f \models_\alpha F_i(x)$ is satisfied.*

Let us note that for every $\overline{\alpha} \preceq \overline{\beta}$ the relation $\Vdash_{\overline{\alpha}}$ is monotone, i.e., if $\tau \subseteq \rho$ are $\overline{\alpha}$-regular finite parts and $\tau \Vdash_{\overline{\alpha}} F_i(x)$, then $\rho \Vdash_{\overline{\alpha}} F_i(x)$, and also if $\tau \Vdash_{\overline{\alpha}} \neg F_i(x)$, then $\rho \Vdash_{\overline{\alpha}} \neg F_i(x)$.

Lemma 3.12. *Let f be a regular enumeration. Then:*
(1) for every $\overline{\alpha} \preceq \overline{\zeta}$, $f \models_\alpha F_i(x) \Leftrightarrow (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\alpha}}\ \&\ \tau \Vdash_{\overline{\alpha}} F_i(x))$;
(2) for every $\overline{\alpha} \prec \overline{\zeta}$, $f \models_\alpha \neg F_i(x) \Leftrightarrow (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\alpha}}\ \&\ \tau \Vdash_{\overline{\alpha}} \neg F_i(x))$.

Proof. We will use transfinite induction over α. First let $\alpha = 0$. Then the validity of (1) follows from the compactness of the enumeration operators Γ_i. Now let us prove (2). Let $f \models_0 \neg F_i(x)$. In order to obtain a contradiction assume that, for every $\overline{0}$-regular $\tau \subseteq f$, is true that $\tau \nVdash_{\overline{0}} \neg F_i(x)$, i.e., for every $\overline{0}$-regular $\tau \subseteq f$ there is $\rho \in \mathcal{R}_{\overline{0}}$ such that $\rho \supseteq \tau$ and $\rho \Vdash_{\overline{0}} F_i(x)$. Consider the set $S = \{\rho \in \mathcal{R}_{\overline{0}} \mid \rho \Vdash_{\overline{0}} F_i(x)\}$. It is clear that $S \leq_e \mathcal{P}_0$ and therefore there is an index j, for which $S = S_j^{\overline{0}}$. Let $\mu \subseteq f$ a $\overline{1}$-regular finite part such that $|\mu|_{\overline{1}} > j$. Such one exists, because f is regular and $1 \leq \zeta$. According to the definition of the $\overline{1}$-regular finite parts there is a $\overline{0}$-regular finite part $\rho_0 \subseteq \mu$ such that $\rho_0 \in S_j^{\overline{0}} = S$. Then $\rho_0 \subseteq f$ and from (1) $f \models_0 F_i(x)$, which is a contradiction.

Now suppose that (1) and (2) are true for every $\delta < \alpha$. We will show that the assertions are also true for α.

a) $\alpha = \beta + 1$. First we show (1). Let $f \models_\alpha F_i(x)$. Then there is v such that $\langle v, x\rangle \in W_i$ and $(\forall u \in D_v)((u = \langle i_u, x_u, 0\rangle \& f \models_\beta F_{i_u}(x_u)) \vee (u = \langle i_u, x_u, 1\rangle \& f \models_\beta \neg F_{i_u}(x_u)))$. According to the induction hypothesis we obtain $\tau_0, \tau_1 \subseteq f$ such that $(\forall u \in D_v)((u = \langle i_u, x_u, 0\rangle \& \tau_0 \Vdash_{\overline{\beta}} F_{i_u}(x_u)) \vee (u = \langle i_u, x_u, 1\rangle \& \tau_1 \Vdash_{\overline{\beta}} \neg F_{i_u}(x_u)))$. Since one of the finite parts is extending the other and the forcing relation is monotone, we may assume $\tau_0 = \tau_1 = \tau$. Then from the definition of the $\overline{\alpha}$-forcing we obtain that $\tau \Vdash_{\overline{\alpha}} F_i(x)$.

The reverse is analogous.

Let us now prove (2). The reasoning is analogous to that of the case $\alpha = 0$. Let $f \models_\alpha \neg F_i(x)$. In order to obtain a contradiction assume that for every $\overline{\alpha}$-regular $\tau \subseteq f$ is true that $\tau \nVdash_{\overline{\alpha}} \neg F_i(x)$, i.e., for every $\overline{\alpha}$-regular $\tau \subseteq f$ there is $\rho \in \mathcal{R}_{\overline{\alpha}}$ such that $\rho \supseteq \tau$ and $\rho \Vdash_{\overline{\alpha}} F_i(x)$. Consider the set $S = \{\rho \in \mathcal{R}_{\overline{\alpha}} \mid \rho \Vdash_{\overline{\alpha}} F_i(x)\}$. It is clear that $S \leq_e \mathcal{P}_\alpha$ and therefore there is an index j for which $S = S_j^{\overline{\alpha}}$. Let $\mu \subseteq f$ be such an $\overline{\alpha+1}$-regular finite part that $|\mu|_{\overline{\alpha+1}} > j$. Such finite part exists as f is regular and $\alpha + 1 \leq \zeta$. According to the definition of the $\overline{\alpha+1}$-regular finite parts, there is an $\overline{\alpha}$-regular finite part $\rho_0 \subseteq \mu$ such that $\rho_0 \in S_j^{\overline{\alpha}} = S$. Then $\rho_0 \subseteq f$, $\rho_0 \Vdash_{\overline{\alpha} F_i(x)}$ and form (1) we obtain $f \models_\alpha F_i(x)$, which is a contradiction.

The opposite follows directly from (1).

b) $\alpha = \lim \alpha(p)$. First we prove (1). Let $f \models_\alpha F_i(x)$. Then there is a v such that $\langle v, x\rangle \in W_i$ and $(\forall u \in D_v)(u = \langle p_u, i_u, x_u\rangle \& f \models_{\alpha(p_u)} F_{i_u}(x_u))$. Then according to the induction hypothesis, for every $u \in D_v$, $u = \langle p_u, i_u, x_u\rangle$ there is $\tau_u \subseteq f$ such that $\tau_u \Vdash_{\overline{\alpha(p_u)}} F_{i_u}(x_u)$. Since D_v is finite, then there is $\tau \subseteq f$ such that $\tau_u \subseteq \tau$ for all $u \in D_v$. As the forcing is monotone $\tau \Vdash_{\overline{\alpha(p_u)}} F_{i_u}(x_u)$ for every $u \in D_v$. Then according to the definition of the α-forcing $\tau \Vdash_{\overline{\alpha}} F_i(x)$.

Now suppose that there is $\tau \subseteq f$ such that $\tau \Vdash_{\overline{\alpha}} F_i(x)$. Then there is v such that $\langle v, x\rangle \in W_i$ and $(\forall u \in D_v)(u = \langle p_u, i_u, x_u\rangle \& \tau \Vdash_{\overline{\alpha(p_u)}} F_{i_u}(x_u))$. Without loss of generality we may assume that τ is $\overline{\alpha(p_u)}$-regular for every $u \in D_v$. Then according to the induction hypothesis $f \models_{\alpha(p_u)} F_{i_u}(x_u)$ for every $u \in D_v$. Therefore $f \models_\alpha F_i(x)$.

The proof of (2) repeats the proof for the case $\alpha = \beta + 1$. □

Proposition 3.3. *Let f be a regular enumeration. Then f is quasiminimal over B_0, i.e., $B_0 <_e f$ and for every total set X is true that:*

$$X \leq_e f \Longrightarrow X \leq_e B_0.$$

Proof. First let us prove that $B_0 <_e f$. We know from proposition 3.2 that $B_0 \leq_e f$. It remains to show that $f \not\leq_e B_0$. In order to obtain a contradiction assume that $f \leq_e B_0$. Then the set $R = \{\tau \in \mathcal{R}_0 \mid \exists x \exists y (f(x) = y \& f(x) \neq \tau(y))\}$ is e-reducible to B_0. Then there is an index i_0 for which $R = S^0_{i_0}$. As f is regular there is a $\overline{1}$-regular finite part $\tau \subseteq f$ such that $|\tau|_{\overline{1}} > i_0$. According to the definition of the $\overline{1}$-regular finite parts, there is a number l_{i_0} such that $\tau_0 = \tau \restriction l_{i_0}$ either is in $S^0_{i_0}$ or no 0-regular extension of τ_0 is in $S^0_{i_0}$. Since $\tau_0 \subseteq f$ it is clear that the first case is impossible. On the other hand, we may extend τ_0 and obtain the finite part τ_1 in such a way, that $\tau_0 \subseteq \tau_1$ and $\tau_1 \in R$. Therefore the second case is also impossible. Therefore, $f \not\leq_e B_0$.

Let us now prove the second part of the quasiminimality condition.

Let A be a total set such that $A \leq_e f$. Since A is total, then there is a total function ψ such that $\langle\psi\rangle \equiv_e A$. Since $\psi \leq_e f$, then there is an i such that $\langle\psi\rangle = \Gamma_i(\langle f\rangle)$. Now consider the set of 0-regular finite parts

$$S = \{\tau \in \mathcal{R}_0 \mid \exists x \exists y_1 \exists y_2 (y_1 \neq y_2 \;\&\; \tau \Vdash_0 F_i(\langle x, y_1\rangle) \;\&\; \tau \Vdash_0 F_i(\langle x, y_2\rangle))\}$$

The condition selecting the finite parts is r.e. and therefore $S \leq_e B_0$. Then there is a j such that $S = S^{\overline{0}}_j$. Let $\rho \subseteq f$ be a finite part such that $|\rho|_1 \geq j+1$. Such a ρ exists, because f is a regular enumeration. Let $n_0, l_0, b_0, \ldots, n_j, l_j, b_j, \ldots$ be the numbers satisfying the definition of the 1-regular finite parts for ρ. Then $\rho \restriction l_j = \mu_{\overline{0}}\left(\rho \restriction (n_j + 1), S^{\overline{0}}_j\right)$. According to the definition of μ either $\rho \restriction l_j \in S^{\overline{0}}_j$ or none of its 0-regular extensions is in $S^{\overline{0}}_j$. Let us assume that the first holds. Then $\rho \restriction l_j \Vdash_0 \langle x, y_1\rangle$ and $\rho \restriction l_j \Vdash_0 \langle x, y_2\rangle$ for some x and $y_1 \neq y_2$. Then $f \models_0 \langle x, y_1\rangle$ and $f \models_0 \langle x, y_1\rangle$ and therefore $\psi(x) = y_1 \neq y_2 = \psi(x)$ which is not possible. Therefore none of the 0-regular extensions of ρ is in $S^{\overline{0}}_j$.

Now consider the set

$$S' = \left\{ \tau \in \mathcal{R}_{\overline{0}} \;\middle|\; \begin{array}{l} (\tau \supseteq \rho \restriction l_j) \;\&\; (\exists \delta_1, \delta_2 \in \mathcal{R}_{\overline{0}})(lh(\rho) \geq lh(\delta_{1/2}) \;\&\\ (\forall z \geq l_j)(\delta_{1/2}(z) \neq \bot \Rightarrow \rho(z) = \bot) \;\&\\ \exists x \exists y_1 \exists y_2 (y_1 \neq y_2 \& \delta_1 \Vdash_{\overline{0}} F_i(\langle x, y_1 \rangle) \;\&\; \delta_2 \Vdash_{\overline{0}} F_i(\langle x, y_2 \rangle)))\end{array} \right\}$$

As above, $S' = S^{\overline{0}}_{j'}$ for some j' and there is a finite part $\tau_0 \subseteq f$ such that either $\tau_0 \in S^{\overline{0}}_{j'}$ or no 0-regular extension of τ_0 is in $S^{\overline{0}}_{j}$. Let us assume that the first one holds and let $\delta_1, \delta_2, x, y_1, y_2$ satisfy the condition. As ψ is a total function, $\psi(x) = y$ for some y. Without loss of generality we may assume $y \neq y_1$. Then there is a 0-regular finite part $\tau_1 \subseteq f$ such that $\tau_1 \supseteq \tau_0$ and $\tau_1 \Vdash_0 F_i(\langle x, y \rangle)$. Therefore $\text{lh}(\tau_1) \geq \text{lh}(\delta_1)$ and $\delta_1(z) \neq \bot \Rightarrow \tau_1(z) = \bot$. The last one guarantees the existence of a finite part τ_1' such that $\langle \tau_1' \rangle = \langle \tau_1 \rangle \cup \langle \delta_1 \rangle$. Then $\tau_1' \supseteq \rho \restriction l_j$ and $\tau_1' \Vdash_{\overline{0}} F_i(\langle x, y \rangle)$, and $\tau_1' \Vdash_0 F_i(\langle x, y_1 \rangle)$. Therefore $\tau_1' \in S$ which contradicts the property of $\rho \restriction l_j$. Thus none of the 0-regular extensions of τ_0 is in $S^{\overline{0}}_{j'}$.

Finally consider the set

$$R = \{\tau \in \mathcal{R}_{\overline{0}} \mid \tau \supseteq \tau_0\}.$$

It is clear that $R \leq_e B_0$. All 0-regular finite subparts of f are in R and therefore $\langle \psi \rangle \subseteq \{\langle x, y \rangle \mid (\exists \tau \in R)(\tau \Vdash_0 F_i(\langle x, y \rangle))\}$. For every two finite parts $\rho_1, \rho_2 \in R$ if $\rho_1 \Vdash_{\overline{0}} F_i(\langle x, y_1 \rangle)$ and $\rho_2 \Vdash_{\overline{0}} F_i(\langle x, y_2 \rangle)$, then $y_1 = y_2$. In the contrary case the $\overline{0}$-regular extension τ_1 of τ_0 having the property $\text{lh}(\tau_1) = \max\{\text{lh}(\rho_1), \text{lh}(\rho_2)\}$ and $(\forall z \geq \text{lh}(\tau_0))(\tau_2(z) = \bot)$ is in S'. But this contradicts the property of τ_0 which was proved above. Then $\{\langle x, y \rangle \mid (\exists \tau \in R)(\tau \Vdash_0 F_i(\langle x, y \rangle))\} \subseteq \langle \psi \rangle$ and therefore these two sets coincide. But $\{\langle x, y \rangle \mid (\exists \tau \in R)(\tau \Vdash_0 F_i(\langle x, y \rangle))\} \leq_e B_0$ and therefore $\langle \psi \rangle \leq_e B_0$. □

Proposition 3.4. *Let f be a regular enumeration and $\alpha \leq \zeta$. Then the following assertions hold:*

(1) *if $\alpha = \beta + 1$, then $f^{(\alpha)} \leq_e f^+ \oplus \mathcal{P}'_\alpha$;*

(2) *if α is a limit ordinal then $f^{(\alpha)} \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$.*

Proof. First let $\alpha = \beta + 1$. Recall that $f^{(\alpha)} = L^+_{f^{(\beta)}}$, where $L_{f^{(\beta)}} = \{\langle y, z \rangle \mid y \in \Gamma_z(f^{(\beta)})\}$. There is a z_0 not depending on β such that $L_{f^{(\beta)}} = \Gamma_{z_0}(f^{(\beta)})$. Therefore

$$f \models_\beta F_{h(\beta, z_0)}(x) \iff x \in L_{f^{(\beta)}}.$$

Now applying Lemma 3.12, we obtain

$$x \in L_{f^{(\alpha)}} \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\beta}} \;\&\; \tau \Vdash_{\overline{\beta}} F_{h(\beta, z_0)}(x)),$$

$$x \in \mathbf{N} \setminus L_{f^{(\beta)}} \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\beta}} \;\&\; \tau \Vdash_{\overline{\beta}} \neg F_{h(\beta, z_0)}(x)).$$

Therefore, according to Proposition 3.1, and as the condition $\tau \subseteq f$ is uniformly recursive in f^+, we obtain that $L_{f^{(\beta)}}$ and $\mathbf{N} \setminus L_{f^{(\beta)}}$ are uniformly e-reducible $f^+ \oplus \mathcal{P}'_\beta$. Therefore $f^{(\alpha)} \leq_e f^+ \oplus \mathcal{P}'_\beta$.

Now let α be a limit ordinal. Then there is a z_0 not depending on α, such that $f^{(\alpha)} = \Gamma_{z_0}(f^{(\alpha)})$. Therefore

$$x \in f^{(\alpha)} \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{\overline{\alpha}} \ \& \ \tau \Vdash_{\overline{\alpha}} F_{h(\alpha,z_0)}).$$

According to Proposition 3.1 we obtain $f^{(\alpha)} \leq f^+ \oplus \mathcal{P}_\alpha$. According to Proposition 3.2, $\mathcal{P}_\alpha \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$. Therefore $f^{(\alpha)} \leq_e f^+ \oplus \mathcal{P}_{<\alpha}$. □

From Proposition 3.2 and 3.4 we obtain the following

Corollary 3.5. *Let f be a regular enumeration and let $\alpha \leq \zeta$. Then:*
(1) if $\alpha = \beta + 1$, then $f^{(\alpha)} \equiv_e f^+ \oplus \mathcal{P}'_\beta$;
(2) if α is a limit ordinal, then $f^{(\alpha)} \equiv_e f^+ \oplus \mathcal{P}_{<\alpha}$.

The following two definitions will be helpful in proving the existence of regular enumerations.

Let us fix a total function σ, such that for every $\alpha \leq \zeta$ $\sigma(\alpha) \in B_\alpha$.

Definition 3.8. Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an approximation of α. We say that τ is $\overline{\alpha}$-complete for σ if

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \Rightarrow \sigma(\beta) \in \tau(B^{\tau}_{\overline{\beta}}).$$

Now let us fix a sequence of sets of natural numbers $\{A_\gamma\}_{\gamma<\zeta}$ such that $(\forall \gamma < \zeta)(A_\gamma \not\leq_e \mathcal{P}_\gamma)$.

Definition 3.9. let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an approximation of α. We say that the finite part τ is $\overline{\alpha}$-omitting in respect to $\{A_\gamma\}$ iff for every $\overline{\beta} \in Reg(\tau, \overline{\alpha})$ the following is true:

If $\beta = \delta + 1$, $\overline{\delta}$ is the δ predecessor of $\overline{\beta}$ and $|\tau|_{\overline{\beta}} = r + 1$, then for every $p \leq r$ there exist a $q_p \in$ dom (τ) and a $\overline{\delta}$-regular finite part $\rho_{p+1} \subseteq \tau$ such that

a) $\rho_{p+1} \Vdash_{\overline{\delta}} F_p(q_p) \ \& \ \tau(q_p) \notin A_\delta$;
b) $\rho_{p+1} \Vdash_{\overline{\delta}} \neg F_p(q_p) \ \& \ \tau(q_p) \in A_\delta$.

Note, that, as for all x the assertion $x \in A_\delta \vee x \notin A_\delta$ holds, then the conditions *a)* and *b)* are equivalent to

a′) $\tau(q_p) \notin A_\delta \Longrightarrow \rho_{p+1} \Vdash_{\overline{\delta}} F_p(q_p)$;
b′) $\tau(q_p) \in A_\delta \Longrightarrow \rho_{p+1} \Vdash_{\overline{\delta}} \neg F_p(q_p)$.

If $\overline{\delta} = \langle \delta_0, \delta_1, \ldots, \delta \rangle$ is an approximation of δ and $\delta < \alpha$, then we will note the approximation $\langle \delta_0, \delta_1, \ldots, \delta, \alpha \rangle$ of α with $\langle \overline{\delta}, \alpha \rangle$.

Now we are ready to prove that the regular enumerations exist.

Proposition 3.5. *Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an approximation of α. Then the following assertions hold:*

(1) For every $\overline{\alpha}$-regular finite part τ and every $y \in \mathbf{N}$ there is a $\overline{\alpha}$-regular extension ρ of τ such that $|\rho|_{\overline{\alpha}} = |\tau|_{\overline{\alpha}} + 1$, $\rho(\mathrm{lh}(\tau)) = y$, ρ is $\overline{\alpha}$-omittimg and $\overline{\alpha}$-complete.

(2) For every $\overline{\delta} \prec \overline{\alpha}$, for every $\overline{\delta}$-regular τ of rank 1 and every $y \in \mathbf{N}$ there is a $\overline{\delta}, \alpha$-regular extension ρ of τ of rank 1 such that $\rho(\mathrm{lh}(\tau)) = y$, ρ is $\overline{\delta}, \alpha$-omitting and $\overline{\delta}, \alpha$-complete.

Proof. We will prove simultaneously (1) and (2) with transfinite induction over α.

a) $\alpha = 0$. In this case (2) is trivial. Now let us consider (1). Let τ be 0-regular finite part and let $y \in \mathbf{N}$. Set ρ to be

$$\rho(x) = \begin{cases} \tau(x), & x < lh(\tau) \\ y, & x = lh(\tau) \\ \sigma(0), & x = lh(\tau) + 1 \\ \neg!, & x > lh(\tau) + 1 \end{cases}$$

Then ρ is a 0-regular finite part satisfying all the desired properties.

b) Let $\alpha = \beta + 1$ and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. First we prove (1).

Let τ be $\overline{\alpha}$-regular finite part and let $y \in \mathbf{N}$. Let also dom $(\tau) = [0, q-1]$ and $|\tau|_{\overline{\alpha}} = r + 1$. Note, that according to the induction hypothesis for (1), it is true that for every $\overline{\beta}$-regular finite part θ, every set $Z \subseteq \mathcal{R}_{\overline{\beta}}$ and every $y \in \mathbf{N}$ the function $\mu_{\overline{\beta}}(\theta * y, Z)$ has a value. Let us denote n_{r+1} with q. As τ is $\overline{\beta}$-regular, then $\rho' = \mu_{\overline{\beta}}(\tau * y, S^{\overline{\beta}}_{r+1})$ is defined. Then let $l_{r+1} = lh(\rho')$. We will construct a special $\overline{\beta}$-regular $r+1$-omitting extension of ρ'. We will define with induction over $p \leq r+2$ the $\overline{\beta}$-regular finite parts ρ_p and the numbers q_p. Set $q_0 = l_{r+1}$ and $\rho_0 = \rho'$. Assume that for some $p < r+2$ the number q_p and the finite part ρ_p are defined. Consider the set

$$C = \{x \mid (\exists \rho \supseteq \rho_p)(\rho \in \mathcal{R}_{\overline{\beta}} \;\&\; \rho(q_p) = x \;\&\; \rho \Vdash_{\overline{\beta}} F_p(q_p)\}.$$

Note that

$$x \notin C \iff (\forall \rho \in \mathcal{R}_{\overline{\beta}})(\rho \supseteq (\rho_p * x) \Longrightarrow \rho \nVdash_{\overline{\beta}} F_p(q_p)).$$

From the definition of C and Proposition 3.1 we obtain $C \leq_e \mathcal{P}_\beta$ and therefore $C \neq A_\beta$. Let x_0 be the least number such that

$$x_0 \in A_\beta \,\&\, x_0 \notin C \;\vee\; x_0 \notin A_\beta \,\&\, x_0 \in C.$$

Then set $\rho_{p+1} = \mu_{\overline{\beta}}\left(\rho_p * x_0, X^{\overline{\beta}}_{\langle p, q_p \rangle}\right)$ and $q_{p+1} = \mathrm{lh}(\rho_{p+1})$.

Now we obtain that $\rho'' = \rho_{r+2}$ is a $\overline{\beta}$-regular $r+1$-omitting extension ρ_0. Set $b_{r+1} = \mathrm{lh}(\rho'')$. Finally set ρ to be a $\overline{\beta}$-regular extension of ρ'', such that

$|\rho|_{\overline{\beta}} = |\rho''|_{\overline{\beta}} + 1$, $\rho(b_{r+1}) = \sigma(\alpha)$, ρ is a $\overline{\beta}$-omitting and $\overline{\beta}$-complete. Then ρ satisfies (1) from the theorem. Indeed, from the construction of ρ we obtain that ρ is an $\overline{\alpha}$-regular extension of $\tau * y$ and $|\rho|_{\overline{\alpha}} = |\tau|_{\overline{\alpha}} + 1$. In order to show that ρ is $\overline{\alpha}$-complete in respect to σ recall that according to Lemma 3.8

$$\overline{\delta} \in Reg(\rho, \overline{\alpha}) \iff \overline{\delta} = \overline{\alpha} \ \vee \ \overline{\delta} \in Reg(\rho, \overline{\beta}).$$

Now fix a $\overline{\delta} \in Reg(\rho, \overline{\alpha})$. If $\overline{\delta} = \overline{\alpha}$ (i.e., $\delta = \alpha$) then $\sigma(\alpha) = \rho(b_{r+1})$. If $\delta \in Reg(\rho, \overline{\beta})$, then, since ρ is $\overline{\beta}$-complete finite part, there is a $b_\delta \in dom(\rho)$, such that $\sigma(\delta) = \rho(b_\delta)$. Therefore ρ is $\overline{\alpha}$-complete.

Now let us prove that ρ is $\overline{\alpha}$-omitting. Fix $\overline{\delta + 1} \in Reg(\rho, \overline{\alpha})$. Then again according to Lemma 3.8 either $\overline{\delta} = \overline{\beta}$ or $\overline{\delta + 1} \in Reg(\rho, \overline{\beta})$ holds. First let $\delta = \beta$. Then as $|\rho|_{\overline{\alpha}} = r + 2$, fix a $p \leq r + 1$. Consider the finite part ρ_{p+1} and the number q_p from the construction. If $\rho_{p+1}(q_p) \in A_\beta$, it follows from the construction that $\rho_{p+1}(q_p)$ is not in the corresponding set C. Now according to the note made after the definition of C, we have $\rho_{p+1} \Vdash_{\overline{\beta}} \neg F_p(q_p)$. Therefore the condition (a') from the definition of the $\overline{\alpha}$-omitting holds. On the other hand, if $\rho_{p+1}(q_p) \notin A_\beta$ holds, then ρ_{p+1} is the least $\overline{\beta}$-regular extension of $\rho_p * (\rho_{p+1}(q_p))$ such that $\rho_{p+1} \Vdash_{\overline{\beta}} F_p(q_p)$ and there for the condition (b') from the definition of the $\overline{\alpha}$-omitting is satisfied.

If $\overline{\delta + 1} \in Reg(\rho, \overline{\beta})$, then we obtain the omitting conditions from the fact that ρ is a $\overline{\beta}$-omitting finite part.

Now let us prove (2). Let $\overline{\delta} \prec \overline{\alpha}$ and let τ be a $\overline{\delta}$-regular finite part of rank 1.

1) $\delta = \beta$. Then $\overline{\delta} = \overline{\beta}$ and beside this $\overline{\beta}$ is the β-predecessor of $\overline{\delta}, \alpha$. Let $n_0 = \mathrm{lh}(\tau)$ and $\rho_0 = \mu_{\overline{\beta}}\left(\tau * y, S_0^{\overline{\beta}}\right)$. Let also ρ_1 be a 0-omitting, $\overline{\beta}$-regular extension of ρ_0, built as above, let $b_1 = \mathrm{lh}(\rho_1)$, and let ρ be a $\overline{\beta}$-complete, $\overline{\beta}$-omitting extension of ρ_1, such that $\rho_1(b_1) = \sigma(\alpha)$ and $|\rho|_{\overline{\beta}} = |\rho_1|_{\overline{\beta}} + 1$. It is clear that ρ is a $\langle\overline{\delta}, \alpha\rangle$-regular finite part with rank 1, which is α-complete and α-omitting.

2) $\delta < \beta$. Then according to Lemma 3.2 the β-predecessor of $\langle\overline{\delta}, \alpha\rangle$ is $\langle\overline{\delta}, \beta\rangle$ and $\overline{\delta} \prec \overline{\beta}$ holds. Using the induction hypothesis extend τ to a $\langle\overline{\delta}, \beta\rangle$-regular finite part ρ_1 of rank 1, such that $\rho_1(\mathrm{lh}(\tau)) = y$. Then we extend ρ_1 to a $\langle\overline{\delta}, \alpha\rangle$-complete and $\langle\overline{\delta}, \alpha\rangle$-omitting finite part ρ of rank 1 as in the prove of (1).

c) Let $\alpha = \lim \alpha(p)$. Let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha\rangle$ and let $p_0 = \mu p[\alpha_n < \alpha(p)]$. As in the previous case, let us first prove (1).

Let τ be an α-regular finite part with rank $r + 1$ and let $y \in \mathbf{N}$. It is clear that τ is an $\overline{\alpha(p_0 + 2r + 1)}$-regular finite part with rank 1. According to the induction hypothesis for (2) there is an $\langle\overline{\alpha(p_0 + 2r + 1)}, \alpha(p_0 + 2r + 2)\rangle$-regular extension ρ_0 of τ of rank 1 such that $\rho_0(\mathrm{lh}(\tau)) = y$. Set $b_{r+1} = \mathrm{lh}(\rho_0)$. Again, according to the induction hypothesis for (2), we construct a $\langle\overline{\alpha(p_0 + 2r + 1)}, \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3)\rangle$-regular extension ρ of ρ_0 of rank 1, such that $\rho(b_{r+1}) = \sigma(\alpha)$ and ρ is $\langle\overline{\alpha(p_0 + 2r + 1)}, \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3)\rangle$-complete and $\langle\overline{\alpha(p_0 + 2r + 1)}, \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3)\rangle$-omitting. Note that $\langle\overline{\alpha(p_0 + 2r + 1)}, \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3)\rangle = \overline{\alpha(p_0 + 2r + 3)}$. Therefore ρ is an $\overline{\alpha}$-regular finite part of rank $r + 2$. It remains to show that ρ is $\overline{\alpha}$-complete and $\overline{\alpha}$-omitting. Let $\overline{\beta} \in Reg(\rho, \overline{\alpha})$. Then

$\overline{\beta} = \overline{\alpha}$ or $\overline{\beta} \in Reg(\tau, \overline{\alpha(p_0 + 2r + 3)})$. In both cases it follows from the construction that $\sigma(\beta) \in \rho(B^{\rho}_{\overline{\beta}})$.

In order to show, that ρ is $\overline{\alpha}$-omitting, let us assume that $\beta = \delta + 1$. Then $\beta \neq \alpha$ and therefore $\overline{\beta} \in Reg(\tau, \overline{\alpha(p_0 + 2r + 3)})$. As ρ is $\overline{\alpha(p_0 + 2r + 3)}$-omitting then it satisfies the omitting conditions in respect to β.

Finally let us show (2). Let $\overline{\delta} \prec \overline{\alpha}$ and let τ be a $\overline{\delta}$-regular finite part with rank 1. Let $y \in \mathbf{N}$ and let also $p_\delta = \mu p[\delta < \alpha(p)]$. According to the induction hypothesis for (2), there is a $\langle \overline{\delta}, \alpha(p_\delta)\rangle$-regular extension ρ_1 of τ such that $\rho_1(\mathrm{lh}(\tau)) = y$ and ρ_1 has $\langle \overline{\delta}, \alpha(p_\delta)\rangle$-rank 1. Then again according to the induction hypothesis for (2) we obtain a $\langle \overline{\delta}, \alpha(p_\delta), \alpha(p_\delta + 1)\rangle$-regular extension ρ of ρ_1, which has rank 1 and for which $\rho(b_0) = \sigma(\alpha)$ holds and which also is $\langle \overline{\delta}, \alpha(p_\delta), \alpha(p_\delta + 1)\rangle$-complete and $\langle \overline{\delta}, \alpha(p_\delta), \alpha(p_\delta + 1)\rangle$-omitting. Then ρ is $\langle \overline{\delta}, \alpha\rangle$-regular extension of τ with rank 1 which is $\langle \overline{\delta}, \alpha\rangle$-complete and $\langle \overline{\delta}, \alpha\rangle$-omitting. □

Note that from the proof we have that the construction is recursive in the set

$$(\bigoplus_{\gamma < \zeta} A^+_\gamma) \oplus \sigma \oplus \mathcal{P}_\alpha.$$

Now we are ready to prove the main theorem.

Proof of Theorem 1.1. Let us fix an arbitrary approximation $\overline{\zeta}$ of ζ. We will construct recursively in Q a sequence of finite regular parts $\{\tau_s\}$ such that $\tau_s \subseteq \tau_{s+1}$ and that the partial function $f = \bigcup_s \tau_s$ is a regular enumeration. Using the previous propositions and some additional reasoning we will see that the set $F \underset{+}{=} \langle f \rangle$ has the desired properties.

As Q is total and $\mathcal{P}_\zeta \leq_e Q$ then according to Lemma 3.2 there are a recursive in Q function $\sigma(\gamma, i)$, such that for every $\gamma \leq \zeta$ the function $\lambda i.\sigma(\gamma, i)$ is enumerating B_γ. Let us fix σ. When constructing the sequence $\{\tau_s\}$, we will ensure that every finite part τ_s is $\overline{\zeta}$-regular of $\overline{\zeta}$-rank equal to $s + 1$, and τ_{s+1} is $\overline{\zeta}$-omitting in respect to $\{A_\gamma\}$ and $\overline{\zeta}$-complete in respect to $\sigma_s = \lambda\gamma.\sigma(\gamma, (s)_1)$ where $s = \langle (s)_0, (s)_1\rangle$. Let us also fix a recursive in Q enumeration $y_0, y_1, \ldots, y_s, \ldots$ of Q.

We begin by setting τ_0 to be an arbitrary $\overline{\zeta}$-regular finite part with $\overline{\zeta}$-rank 1. Let τ_s be constructed. Then according to Proposition 3.5 we can obtain recursively in Q a $\overline{\zeta}$-regular extension τ_{s+1} of τ_s, such that $\tau_{s+1}(\mathrm{lh}(\tau_s)) = y_s$, $|\tau_{s+1}|_{\overline{\zeta}} = |\tau_s|_{\overline{\zeta}} + 1$ and τ_{s+1} is $\overline{\zeta}$-omitting and $\overline{\zeta}$-complete in respect to σ_s. Note that τ_{s+1} is strictly extending τ_s.

First let us show that f is a regular enumeration.

Note that f is a partial function from $\mathbf{N}$ in $\mathbf{N}$, and for every $\rho \subseteq f$ there is an index s, such that $\rho \subseteq \tau_s$. Then consider $\overline{\gamma} \preceq \overline{\zeta}$ and $z \in B_\gamma$. Let us fix an s such big that every $\overline{\zeta}$-regular finite part of $\overline{\zeta}$-rank at least s is $\overline{\gamma}$-regular (such an s exists according to Lemma 3.2). We can also choose s such that $z = \sigma(\gamma, (s)_1)$ holds. Then as τ_{s+1} is of $\overline{\zeta}$-rank $s + 2$ and is $\overline{\zeta}$-complete in respect to $\sigma_s = \lambda\gamma.\sigma(\gamma, (s)_1)$ we obtain that $z \in \tau_{s+1}(B^{\tau_{s+1}}_{\overline{\gamma}})$. Therefore f is a regular enumeration.

Now we show that $f^{(\zeta)} \equiv_e Q$.

It is clear that $f^+ \leq_e Q$. Beside this as f is regular then, according to Proposition 3.4, $f^{(\zeta)} \leq_e f^+ \oplus \mathcal{P}_\zeta \leq_e Q$. From the proof of Proposition 3.2 we obtain a recursive in $f^+ \oplus \mathcal{P}_\zeta$ procedure which gives us the sequence $q_s = lh(\tau_s)$. It is also true that

$$y \in Q \iff \exists s(y = f(q_s)),$$

and $f(q_s)$ is always defined. Thus $Q \leq_e f^{(\zeta)}$ and therefore $f^{(\zeta)} \equiv_e Q$.

It remains to prove that for every $\gamma < \zeta$, $A_\gamma \not\leq f^{(\gamma)}$ is satisfied.

To obtain a contradiction assume that for some $\gamma < \zeta$, $A_\gamma \leq f^{(\gamma)}$ holds. Then the set $f^{-1}(A_\gamma) = \{x \mid \exists y\, (\langle x, y\rangle \in \langle f \rangle \;\&\; y \in A_\gamma)\}$ is also e-reducible to $f^{(\gamma)}$. Then there is an index i, for which

$$x \in C \iff f \models_\gamma F_i(x).$$

Let $\overline{\gamma + 1}$ be the $\gamma+1$-predecessor of $\overline{\zeta}$ and let $\overline{\gamma}$ be the γ-predecessor of $\overline{\gamma+1}$. Let s be so big that every $\overline{\zeta}$-regular finite part is $\overline{\gamma+1}$-regular of $\overline{\gamma+1}$-rank greater or equal to i (such an s exists according to Lemma 3.2). Then τ_{s+1} is $\overline{\gamma+1}$-regular and $|\tau_{s+1}|_{\overline{\gamma+1}} > i$. As τ_{s+1} is $\overline{\zeta}$-omitting finite part there is a $q \in \operatorname{dom}(\tau_{s+1})$ and a $\overline{\gamma}$-regular finite part $\rho \subseteq \tau_{s+1}$ such that:

$$\rho \Vdash_{\overline{\gamma}} F_i(q) \;\&\; \tau_{s+1}(q) \notin A_\gamma \;\vee\; \rho \Vdash_{\overline{\gamma}} \neg F_i(q) \;\&\; \tau_{s+1}(q) \in A_\gamma.$$

Therefore

$$f(q) \in A_\gamma \Longrightarrow (\exists \rho \subseteq f)(\rho \Vdash_{\overline{\gamma}} F_i(q)) \;\&\; f(q) \notin A_\gamma \Longrightarrow (\exists \rho \subseteq f)(\rho \Vdash_{\overline{\gamma}} \neg F_i(q))$$

Then according to the Truth Lemma (Lemma 3.12),

$$f \models_\gamma F_i(q) \iff q \notin C,$$

which is a contradiction. □

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Received on January 25, 2006

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