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NUMERICAL SOLUTION OF HEAT-CONDUCTION PROBLEMS ON A SEMI-INFINITE STRIP WITH NONLINEAR LOCALIZED FLOW SOURCES [1]

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We consider semi-linear heat problems on a semi-infinite interval. They model systems of temperature regulation in isotropic media with non-uniform source terms which can provide cooling or heating effects. Numerical method for overcoming the nonlinearities in the equation and boundary conditions as well as the unbounded domain is discussed.

Keywords: Parabolic problems, difference schemes, convergence rate, semi-infinite strip, artificial boundary conditions.

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1. INTRODUCTION

This paper is concerned with the following initial-boundary value problems for the one-dimensional heat equation on a semi-infinite interval:

$$(P_1)\qquad \begin{aligned} &u_t - u_{xx} = -F_1(u_x(0,t)), \quad 0 < x < \infty, \quad 0 < t \le T, \quad T < \infty, \\ &u(0,t) = 0, \quad 0 \le t \le T, \\ &u(x,0) = h_1(x), \quad 0 \le x < \infty, \end{aligned}$$

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$$(P_2) \qquad \begin{aligned} &u_t - u_{xx} = \Phi(x).F_2(u_x(0,t),t), \quad 0 < x < \infty, \quad 0 < t \le T, \ T < \infty, \\ &u(0,t) = g(t), \quad 0 \le t \le T, \\ &u(x,0) = h_2(x), \quad 0 \le x < \infty, \end{aligned}$$

where $u = u(x,t)$ denotes the temperature distribution (the unknown), x and t are the spatial and time coordinate respectively; T is a given positive constant. The data functions $h_1(x)$, $h_2(x)$, $g(t)$ (representing initial and boundary conditions) and $\Phi(x)$ are real, defined on R^+. F_1 and F_2 are sink sources of heat energy, uniform in x. Such problems can be thought as by the modelling of a system of temperature regulation in isotropic media, with non-uniform source term $F_1(u_x(0,t))$ for (P_1) and $\Phi(x).F_2(u_x(0,t),t)$ for (P_2), which provides a cooling or heating effect depending upon the properties of F_1 or F_2, related to the source of the localized heat flux $u_x(0,t)$, see [13, 14].

In [3, 15] results on existence, uniqueness and asymptotic behavior of the solution have been proved for problem (P_1). Some results on the behavior of the solution and explicit formula for the solution in special cases are obtained in [13].

The existence, uniqueness and asymptotic behavior of the solution of problem (P_2) are explicated in [14]. Also, the validation of the maximum principle for (P_1) and (P_2) is shown in [13, 14, 15].

The goal of this paper is to solve numerically problems (P_1) and (P_2) with effective and accurate methods.

The remainder part of this work is organized as follows. In Section 2 properties of the solutions to the considered problems are described. The original problems are written in new, equivalent forms, more appropriate for numerical treatment. In the next section we construct exact artificial boundary conditions of the new formulated problems. Also, full discretizations are derived. In Section 4 we present some numerical results, demonstrating the accuracy of the algorithms.

2. PRELIMINARY RESULTS

As observed in [13, 14], the heat flux

$$v(x,t) = u_x(x,t), \tag{2.1}$$

for problems (P_1) and (P_2) satisfies the classical heat conduction problems with a nonlinear convective condition at $x = 0$, which can be written in the forms:

$$(V_1) \qquad \begin{aligned} &v_t - v_{xx} = 0, \quad 0 < x < \infty, \quad 0 < t \le T, \quad T < \infty, \\ &v_x(0,t) = F_1(v(0,t)), \quad 0 \le t \le T, \\ &v(x,0) = h_1'(x), \quad 0 \le x < \infty, \end{aligned}$$

$$(V_2) \qquad \begin{aligned} &v_t - v_{xx} = \Phi'(x).F_2(v(0,t),t), \quad 0 < x < \infty, \quad 0 < t \le T, \quad T < \infty, \\ &v_x(0,t) = g'(t) - \Phi(0).F_2(v(0,t),t), \quad 0 \le t \le T, \\ &v(x,0) = h_2'(x), \quad 0 \le x < \infty, \end{aligned}$$

For problem (V_1), it has been proved in [13] that the maximum principle holds. A qualitative analysis of the problem (V_2) is given in [14]. The authors show that under some assumptions for $F_2(v(0,t),t)$, $g'(t)$, $\Phi'(x)$ and $h_2'(x)$, we have $v(0,t) > 0$ for $t > 0$ and $u_x(x,t) \geq 0$ for $x \geq 0$, $t \geq 0$. Also the monotonicity properties of the solution of the problem (V_2) are proved.

It's worth to note that if $h_1'(x) \geq 0$ in R^+, $v(0,t).F_1(v(0,t)) > 0$, $v(0,t) > 0$, $t \in [0,T]$ (for problem (V_1)) or for problem (V_2): $h_2(x) > 0$, $\Phi(x) \leq 0$, $\Phi'(x) \geq 0$ in R^+, $v(0,t).F_2(v(0,t),t) > 0$, $\forall\, v(0,t) \neq 0$, $\forall\, t > 0$ and $g(t) \geq 0$ or $\lim_{t\to\infty} g(t) = 0$, $\forall\, t > 0$, together with some other hypotheses, then the corresponding solutions of (P_1) or (P_2), $u(x,t) \to 0$, as $t \to \infty$ uniformly for $x \geq 0$, see [14, 15].

In the work done, we restrict our considerations to the case: $h_i'(x) \geq 0$, $\operatorname{supp} h_i'(x) < \infty$, $i = 1,2$, $\Phi'(x) \geq 0$ and $\operatorname{supp} \Phi'(x) < \infty$. The last constraint we shall remove later. Many physical processes lead to models with compact supported initial datum. Such kind of problems are well studied in [4, 5]. From $\operatorname{supp} h_i'(x) < \infty$, $i = 1,2$ follows that there exists L_i, $i = 1,2$ and $0 < L_i < +\infty$: $h_i'(x) = 0$ for $x > L_i$, $i = 1,2$. Then $v(x,t) \to 0$ when $x \to +\infty$, i.e. $v(+\infty,t) = 0$, $\forall\, t > 0$.

3. NUMERICAL METHOD

We focuss our attention to problems (V_1) and (V_2). Having obtained their numerical solutions, it's very easy to find the solutions of (P_1) and (P_2), using (2.1) and the well-known numerical integration formulas (by Trapezoidal rule, for example).

The most widely used methods are the finite element and the finite difference schemes. Since the grids are finite, then on the grid boundary the same type boundary conditions as on the infinity in the differential problem, are often imposed, see for example [1, 2]. This, however, leads to the loss of accuracy, especially in the case, when the solution does not go to zero as $x \to \infty$ or the compact support of the solution become large in time. More accurate are artificial boundary conditions. For linear parabolic problems with linear boundary conditions such results can be found in [6, 16] and for semi-linear one and two-dimensional heat problems, see [7, 10]. Also, the comparison with other methods is available.

Having in mind all those results, our approach will be to use an artificial boundary method. Generally, it means to introduce artificial boundaries, construct exact boundary conditions on the artificial boundaries and reduce the original problem to an equivalent or approximate problem, defined on a bounded domain. In general, the boundary conditions on the artificial boundaries are obtained by considering the exterior problems outside the artificial boundaries.

The idea is employed from [10, 16].

Let supp $h_i'(x) = [0, L_i]$, $i = 1, 2$ and supp $\Phi'(x) \in [0, L_2]$. For computing the numerical solutions of any of the problems (V_1) and (V_2), we introduce an artificial boundary: $\Gamma_i = \{(x,t) | x = l, \ \ l > L_i, \ \ i = 1, 2; \ \ 0 \le t \le T\}$. Then the domain $\Omega = \{(x,t) | 0 \le x < +\infty, \ \ 0 < t \le T\}$ is divided into the bounded part Ω^0 and unbounded part $\Omega^e = \{(x,t) | l < x < +\infty, \ \ 0 < t \le T\}$. On the domain Ω^e, $h_i'(x) \equiv 0$, $i = 1, 2$ and $\Phi'(x) \equiv 0$. We first consider the restriction of the solution of the considered problems on the domain Ω^e (counterpart domain). In this domain, the solutions of both problems (V_1) and (V_2) satisfy one and the same initial-boundary value problem (V^e).

$$(V^e) \qquad \begin{aligned} & v_t - v_{xx} = 0, \quad (x,t) \in \Omega^e, \\ & v(x,0) = 0, \quad l \le x < +\infty, \\ & v(x,t) \to 0, \quad \text{when} \quad x \to +\infty. \end{aligned}$$

If $v(l,t)$ is given, then (V^e) is a properly posed problem. We can get the solution $(v(x,t))$ for given $(v(l,t))$, see [12].

$$v(x,t) = \frac{x - l}{2\sqrt{\pi}} \int_0^t v(l, \lambda)(t - \lambda)^{-\frac{3}{2}} e^{-\frac{(x-l)^2}{4(t-\lambda)}} d\lambda, \tag{3.1}$$

Next, we shall obtain the artificial boundary condition, using (3.1). Setting $\rho = (x - l)/(2\sqrt{t - \lambda})$, then we have

$$v(x,t) = \frac{2}{\sqrt{\pi}} \int_{\frac{x-l}{2\sqrt{t}}}^{\infty} v\left(l, t - \frac{(x-l)^2}{4\rho^2}\right) e^{-\rho^2} d\rho,$$

$$\frac{\partial v(x,t)}{\partial x} = \frac{1}{\sqrt{\pi t}} v(l, 0) e^{-\frac{(x-l)^2}{4t}} + \frac{1}{\sqrt{\pi}} \int_{\frac{x-l}{2\sqrt{t}}}^{\infty} \frac{\partial v}{\partial t} \left(l, t - \frac{(x-l)^2}{4\rho^2}\right) \frac{l - x}{\rho^2} e^{-\rho^2} d\rho.$$

Returning to the variable λ, we get

$$\frac{\partial v(x,t)}{\partial x} = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial v(l, \lambda)}{\partial \lambda} \frac{1}{\sqrt{t - \lambda}} e^{-\frac{(x-l)^2}{4(t-\lambda)}} d\lambda.$$

Taking the limit $x \to +l$, we obtain the following exact boundary condition, satisfied by the solution $v(x,t)$ on the artificial boundary $x = l$.

$$\frac{\partial v(l,t)}{\partial x} = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial v(l, \lambda)}{\partial \lambda} \frac{1}{\sqrt{t - \lambda}} d\lambda, \quad 0 \le t < +\infty. \tag{3.2}$$

Using (3.2), we reduce original problems (V_1) and (V_2) to problems on the bounded domain $\Omega^0 = \{(x,t)|0 < x < l, \ \ 0 \le t \le T\}$.

$$v_t - v_{xx} = \begin{cases} 0, & (x,t) \in \Omega^0, \quad \text{for } (V_1), \\ \Phi'(x).F_2(v(0,t),t), & (x,t) \in \Omega^0, \quad \text{for } (V_2), \end{cases}$$

$$v_x(0,t) = \begin{cases} F_1(v(0,t)), & 0 \le t \le T, \quad \text{for } (V_1), \\ g'(t) - \Phi(0).F_2(v(0,t),t), & 0 \le t \le T, \quad \text{for } (V_2), \end{cases}$$

$$(Ri) \qquad v_x(l,t) = -\frac{1}{\sqrt{\pi}} \int_0^t v_\lambda(l,\lambda) \frac{1}{\sqrt{t-\lambda}} d\lambda, \quad 0 \le t \le T,$$

$$v(x,0) = \begin{cases} h_1'(x), & 0 \le x \le l, \quad \text{for } (V_1), \\ h_2'(x), & 0 \le x \le l, \quad \text{for } (V_2). \end{cases}$$

The solutions of the problems (V_1) and (V_2) in Ω^e can be computed by formula (3.1) for $v(l,t)$ already known.

If supp $\Phi'(x) = \infty$, then the problem (V^e), corresponding to (V_2) becomes

$$(\tilde{V}^e) \qquad \begin{aligned} & v_t - v_{xx} = \Phi'(x).F_2(v(0,t),t), \quad (x,t) \in \Omega^e, \\ & v(x,0) = 0, \quad l \le x < +\infty. \end{aligned}$$

Now, for the solution $v(x,t)$ of problem $(\tilde{V}^e)$ for given $v(l,t)$ we have (see [12]):

$$\begin{aligned} v(x,t) &= \frac{x-l}{2\sqrt{\pi}} \int_0^t v(l,\lambda)(t-\lambda)^{-\frac{3}{2}} e^{-\frac{(x-l)^2}{4(t-\lambda)}} d\lambda \\ &+ \frac{1}{2\sqrt{\pi}} \int_0^t \frac{F_2(v(0,\lambda),\lambda)}{\sqrt{t-\lambda}} \int_0^\infty \Phi'(\xi) \left[e^{-\frac{(x-l-\xi)^2}{4(t-\lambda)}} - e^{-\frac{(x-l+\xi)^2}{4(t-\lambda)}} \right] d\xi d\lambda \qquad (3.3) \end{aligned}$$

As before, applying the technique: 1° change of variable; 2° differentiating with respect to x; 3° returning to the original variable; 4° taking a limit $x \to l$, for the first addend of (3.3) and only 2° and 4° for the second one, we obtain the following artificial boundary condition at $x = l$:

$$\begin{aligned} \frac{\partial v(l,t)}{\partial x} &= -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial v(l,\lambda)}{\partial \lambda} \frac{1}{\sqrt{t-\lambda}} d\lambda \\ & \quad + \frac{1}{2\sqrt{\pi}} \int_0^t \frac{F_2(v(0,\lambda),\lambda)}{(t-\lambda)^{3/2}} \int_0^\infty \xi \Phi'(\xi) e^{-\frac{\xi^2}{4(t-\lambda)}} d\xi d\lambda \qquad (3.4) \end{aligned}$$

Again, the assumption $v(l,0) = 0$ is an essential one.

Let V_h is a piecewise linear finite element space, defined on an uniform mesh with size h in $\bar{D} = [0, l]$: $\bar{\omega}_h = \{x_i,\ x_i = (i-1)h,\ i = 1, 2, ..., N;\ (N-1)h = l\}$. For a discrete function, defined on $\bar{\omega}_h$, we introduce the following norms:

$$||v||_{L_\infty(\bar{\omega}_h)} = \max_{\bar{\omega}_h} |v(x_i)| \ \text{ and } \ ||v||_{L_2(\bar{\omega}_h)} = \left(\sum_{i=1}^{N} h v^2(x_i) \right)^{\frac{1}{2}}.$$

The standard finite element discretization of the problems (Ri), $i = 1, 2$ is to find

$$v^h \in V_h, \quad v^h = \sum_{i=1}^{N} V_i(t) \varphi_i(x),$$

satisfying the weak forms of the problems (R_i).

Now, after doing a mass lumping, we obtain for $V_i = V_i(t)$, $i = 1, ..., N$ and $0 \le t \le T$ the following system of ordinary integro-differential equations

$$\dot{V}_1 = \frac{2}{h} \left[\frac{V_2 - V_1}{h} - \begin{cases} F_1(V_1), & \text{for } (R1) \\ g'(t) - \Phi(0).F_2(V_1, t) - \frac{h}{2}\Phi'(0).F_2(V_1, t), & \text{for } (R2) \end{cases} \right], \tag{3.5}$$

$$\dot{V}_i = \frac{1}{h^2}[V_{i-1} - 2V_i + V_{i+1}] + \begin{cases} 0, & \text{for } (R1) \\ \Phi'(x_i).F_2(V_1, t), & \text{for } (R2) \end{cases}, i = 2, ..., N-1, \tag{3.6}$$

$$\dot{V}_N = -\frac{2}{h} \left[\frac{1}{\sqrt{\pi}} \int_0^t \frac{\dot{V}_N(\lambda)}{\sqrt{t - \lambda}} d\lambda + \frac{V_N - V_{N-1}}{h} \right]. \tag{3.7}$$

In the case supp $\Phi'(x) = \infty$ (concerning the problem (V_2)), using (3.4), the equation (3.7) becomes

$$\begin{aligned} \dot{V}_N &= -\frac{2}{h} \left[\frac{1}{\sqrt{\pi}} \int_0^t \frac{\dot{V}_N(\lambda)}{\sqrt{t-\lambda}} d\lambda + \frac{V_N - V_{N-1}}{h} - \frac{h}{2} \Phi'(0).F_2(V_1, t) \right. \\ &\left. - \frac{1}{2\sqrt{\pi}} \int_0^t \frac{F_2(V_1(\lambda), \lambda)}{(t-\lambda)^{3/2}} \int_0^\infty \xi \Phi'(\xi) e^{-\frac{\xi^2}{4(t-\lambda)}} d\xi d\lambda \right]. \end{aligned} \tag{3.8}$$

Next, in order to obtain the full discretization of (3.5)-(3.7)(or (3.8)) we define an uniform mesh in time:

$$t_n = n\tau, \quad n = 0, 1, ...M, \quad M\tau = T.$$

The following lemma we also need

Lemma 3.1 ([16]). *Suppose $f(t) \in C^2[0, t_n]$. Then*

$$\left| \int_0^{t_n} \frac{f'(t)dt}{\sqrt{t_n - t}} - \sum_{k=1}^{n} \frac{f(t_k) - f(t_{k-1})}{\tau} \int_{t_{k-1}}^{t_k} \frac{dt}{\sqrt{t_n - t}} \right| \le \frac{(10\sqrt{2} - 11)}{6} \max_{0 \le t \le t_n} |f''(t)| \, \tau^{\frac{3}{2}}.$$

Lemma 3.1 ground on the approximations of the integrals in the formulas (3.7) and (3.8). The integrals (obtained applying this lemma) are calculated exactly. This semi-analytical integration rule has better accuracy and stability properties than the Trapezoidal rule yet involves about the same computational effort. This is the best possible integration rule, since no additional information on $V_N(\lambda)$ is available in the interval $[t_{n-1}, t_n]$, see [11].

Consequently we obtain the full discretization of the system (3.5)-(3.7), $V_i^n = v^h(x_i, t_n)$, $i = 2, ..., N-1$, $n = 1, ..., M$

$$\left[1+\frac{2\tau}{h^2}\right] V_1^n - \frac{2\tau}{h^2} V_2^n +$$

$$\frac{2\tau}{h}\left\{\begin{array}{lr} F_1(V_1^n), & \text{for } (R1) \\ g'(t_n) - \Phi(0).F_2(V_1^n, t_n) - \frac{h}{2}\Phi'(0).F_2(V_1^n, t_n), & \text{for } (R2) \end{array}\right\} = V_1^{n-1},$$

$$\left(1+\frac{2\tau}{h^2}\right) V_i^n - \frac{\tau}{h^2}\left(V_{i-1}^n + V_{i+1}^n\right) - \left\{\begin{array}{lr} 0, & (R1) \\ \tau\Phi'(x_i).F_2(V_1^n, t_n), & (R2) \end{array}\right\} = V_i^{n-1},$$

$$\left(\frac{h}{2}+\frac{2\sqrt{\tau}}{\sqrt{\pi}}+\frac{\tau}{h}\right) V_N^n - \frac{\tau}{h} V_{N-1}^n = \left(\frac{h}{2}+\frac{2\sqrt{\tau}}{\sqrt{\pi}}\right) V_N^{n-1}$$

$$-\frac{2}{\sqrt{\pi}}\sum_{k=1}^{n-1}\left(\sqrt{t_n - t_{k-1}} - \sqrt{t_n - t_k}\right)\left(V_N^k - V_N^{k-1}\right). \tag{3.9}$$

To obtain the full discretization of (3.8) we need, in addition to (3.9) (approximation of the first addend of (3.8)), the approximation of the second addend of (3.8)

Lemma 3.2 ([16]). *Let $f(t) \in C^2[0, t_n]$ and $g(t) = (t_n - t)^{-3/2} e^{-\frac{a^2}{4(t_n - t)}}$ with $a > 0$. Then*

$$\left|\int_0^{t_n} f(t)g(t)dt - \sum_{k=1}^{n}\frac{1}{2}\left[f(t_k) + f(t_{k-1})\right]\int_{t_{k-1}}^{t_k} g(t)dt\right|$$

$$\leq \left(\frac{2c}{a^3}\max_{0\leq t\leq t_n}|f'(t)| + \frac{\sqrt{\pi}}{4a}\max_{0\leq t\leq t_n}|f''(t)|\right)\tau^2, \text{ where } c = \int_0^\infty \left|\frac{3}{2} - \mu^2\right| \mu^2 e^{-\mu^2} d\mu.$$

Calculation of the integrals $\int_{t_{k-1}}^{t_k} g(t)dt$ exactly, leads to the semi-analytical integration approximation, which was commented earlier. Lemma 3.2 is essential for deriving the following discretization:

$$\frac{1}{2\sqrt{\pi}}\int_0^t \frac{F_2(V_1(\lambda), \lambda)}{(t-\lambda)^{3/2}}\int_0^\infty \xi\Phi'(\xi) e^{-\frac{\xi^2}{4(t-\lambda)}} d\xi d\lambda$$

$$\approx h\sum_{i=1}^{\infty}\Phi'(x_i)\sum_{k=1}^{n}\frac{F_2(V_1(t_{k-1}),t_{k-1})+F_2(V_1(t_k),t_k)}{2}\cdot[I(t_k)-I(t_{k-1})], \quad (3.10)$$

where $\quad I(t_k)=\text{erf}\left(\frac{x_i}{2\sqrt{t_n-t_k}}\right)$ and $\quad \text{erf}\,(x)=\frac{2}{\sqrt{\pi}}\int\limits_0^x e^{-\rho^2}d\rho.$

The calculations are tedious, but standard, and we shall outline the main steps:

1. Use Lemma 3.1 for integral with respect to time;
2. Rearrange the integrals and integrand functions;
3. Change the variable λ: $\rho=\frac{\xi}{2\sqrt{t_n-\lambda}}$;
4. Involve the erf integrals;
5. Approximate the integral $\int\limits_0^\infty$ by Rectangular (Trapezoidal or some other quadrature is also possible) rule. At this stage, using Lemma 3.1 would lead to unduly complication of the approximation. Moreover, in contrast to F_2, the integrand function Φ' is a known function of the known argument, thus the usage of Lemma 3.1 can be avoided.

Finally, the full discretization of (3.8) is

$$\left(\frac{h}{2}+\frac{2\sqrt{\tau}}{\sqrt{\pi}}+\frac{\tau}{h}\right)V_n^N-\frac{\tau}{h}V_{n-1}^N-\frac{h\tau}{2}F_2(V_1^n,t_n)\left[\Phi'(0)+\sum_{i=1}^{\infty}\Phi'(x_i)\left(1-erf\left(\frac{x_i}{2\sqrt{\tau}}\right)\right)\right]$$

$$=\left(\frac{h}{2}+\frac{2\sqrt{\tau}}{\sqrt{\pi}}\right)V_{n-1}^N-\frac{2}{\sqrt{\pi}}\sum_{k=1}^{n-1}(V_N^k-V_n^{k-1})(\sqrt{t_n-t_{k-1}}-\sqrt{t_n-t_k})$$

$$+\frac{h\tau}{2}\sum_{i=1}^{\infty}\Phi'(x_i)\left[F_2(V_1^{n-1},t_{n-1})\left(1-erf\left(\frac{x_i}{2\sqrt{\tau}}\right)\right)\right.$$

$$\left.+\sum_{k=1}^{n-1}\left[F_2(V_1^k,t_k)+F_2(V_1^{k-1},t_{k-1})\right][I(t_k)-I(t_{k-1})]\right].$$

For numerical implementations, the infinite series is truncated at a large number of terms, say S, depending on the function $\Phi'(x)$.

The solutions in counterpart domain ($x>l$) can be computed, using integral formulas

$$v(x,t_n)=\frac{1}{2}\sum_{k=1}^{n}\left[erf\left(\frac{x-l}{2\sqrt{t_n-t_{k-1}}}\right)-erf\left(\frac{x-l}{2\sqrt{t_n-t_k}}\right)\right](V_N^k-V_N^{k-1})$$

$$+\begin{cases}0, & \text{for problems } (V_1) \text{ and } (V_2) \text{ with } \operatorname{supp}\Phi'(x)<\infty,\\ A, & \text{for problem } (V_2) \text{ with } \operatorname{supp}\Phi'(x)=\infty,\end{cases}$$

where A is the second addend of (3.3), if it can be calculated exactly, otherwise we use it's approximation, obtained in the similar way as (3.10). We trace the main points of the calculations:

1. Present as a difference of two integrals (the difference between two integrals is only in the exponential power). Both integrals we treat in mush the same manner;
2. Apply Lemma 3.2;
3. Rearrange the integrals and integrand functions;
4. Apply Green formula in order to generate the term $(t-\lambda)^{-3/2}$.

Next, we follow the same steps as before, but the change of the variable λ is $\rho = \frac{x-t\pm\xi}{2\sqrt{t_n-\lambda}}$ for the corresponding integrals. Thus we obtain:

$$A = \frac{1}{2\sqrt{\pi}} \sum_{i=1}^{S} \Phi'(x_i) \sum_{k=1}^{n} \frac{F_2(V_1(t_{k-1}), t_{k-1}) + F_2(V_1(t_k), t_k)}{2} \cdot \bar{I}(t_{k-1}, t_k),$$

$$\bar{I}(t_{k-1}, t_k) = I^+(t_{k-1}, t_k) - I^-(t_{k-1}, t_k), \quad P^{\pm}(t_s) = \frac{(x-l\pm x_i)^2}{4(t_n - t_s)},$$

$$I^{\pm}(t_{k-1}, t_k) = 2\sqrt{t_n - t_{k-1}}e^{-P^{\pm}(t_{k-1})} - 2\sqrt{t_n - t_k}e^{-P^{\pm}(t_k)}$$
$$+\frac{(x-l\pm x_i)\sqrt{\pi}}{2}\left\{\mathrm{erf}\left(\sqrt{P^{\pm}(t_{k-1})}\right) - \mathrm{erf}\left(\sqrt{P^{\pm}(t_k)}\right)\right\}.$$

Remark 3.1. The discrete maximum principle and convergence of the numerical schemes can be proved, as it's already done for similar problems in our previous works [8, 9].

4. NUMERICAL EXAMPLES

In this section we verify numerically the efficiency, convergency and accuracy of the algorithm, based on the construction of artificial boundary condition (ABCM). The results are compared with ones, obtained by standard method: solving numerically the original problem on a large enough finite interval $\bar{D} \in [0, L]$ and imposing zero Dirichlet boundary condition on the remote boundary.

Example 1. The test problem is (P_2), $\Phi(x) = -e^{-x}$, $F_2(u_x(0,t),t) = -u_x(0,t) - \frac{2e^{-\frac{1}{t^2}}}{t^2\sqrt{\pi}}$, $g(t) = \mathrm{erfc}(\frac{1}{t})$ and $h_2(x) \equiv 0$. Then, the exact solution is $u(x,t) = e^{-x}\mathrm{erfc}(\frac{1}{t})$. This example is favourable for standard method, because the solution goes to zero ($x \to \infty$, $t \to \infty$) rapidly. On the other hand, since $v(x, 0) \equiv 0$ in the equivalent problem (V_2), we may take any line $x = l > 0$ as the artificial boundary. Let $l = 1$, $S = 20$ and the ratio $\frac{\tau}{h^2} = 1$ is fixed. In Table 1 we present the errors under different discrete norms, convergence rates and CPU times (in seconds) of the algorithms - ABCM and standard method at $t = 0.5$. The errors are defined as follows:

$$E^h_\infty = \|u - U\|_{L_\infty(\bar{\omega}_h)} \quad \text{and} \quad E^h_2 = \|u - U\|_{L_2(\bar{\omega}_h)}.$$

The convergence rate is computed, using the formula

$$CR = \log_2 \frac{E^h_{2(or\ \infty)}}{E^{\frac{h}{2}}_{2(or\ \infty)}}.$$

TABLE 1. Errors in different norms, convergence rates (CR) and CPU times (sec)

	ABCM, $l=1$, $L=5$		**Standard Method**, $u(L,t)=0$			
h **CR** **CPU**	E^h_∞	E^h_2	$L=5$		$L=10$	
			E^h_∞	E^h_2	E^h_∞	E^h_2
0.1	7.23721e-4	6.923436e-4	7.96384e-4	7.43435e-4	7.53891e-4	7.02529e-4
CPU	*3.3441*		*0.5940*		*0.9372*	
0.05	1.97566e-4	1.87719e-4	2.41093e-4	2.20404e-4	2.11617e-4	1.95553e-4
CR	**1.8731**	**1.8829**	**1.7238**	**1.7541**	**1.8329**	**1.8450**
CPU	*48.3750*		*3.4380*		*5.7970*	
0.025	5.31497e-5	4.96847e-5	8.21020e-5	7.45691e-5	6.70987e-5	6.14957e-5
CR	**1.8942**	**1.9177**	**1.5541**	**1.5635**	**1.6571**	**1.6690**
CPU	*6.5449e+2*		*25.6561*		*41.7352*	
0.0125	1.37550e-5	1.26190e-5	3.95401e-5	3.57905e-5	2.45255e-5	2.18145e-5
CR	**1.9501**	**1.9772**	**1.0540**	**1.0590**	**1.4520**	**1.4951**
CPU	4.5971e+3		*1.9002e+2*		*3.4294e+2*	
0.00625	3.47663e-6	3.17846e-6	3.55344e-5	3.21402e-5	9.80292e-6	8.58736e-6
CR	**1.9842**	**1.9892**	**0.1541**	**0.1552**	**1.3230**	**1.3452**
CPU	7.6963e+4		*1.3051e+3*		*2.9707e+3*	

Even for a fast vanished solution, L is not large enough and we "lose" convergence of the standard method. The reason is that the main source of error: Dirichlet boundary condition, $u(L,t)=0$, remain one and the same, independently of the mesh step size. If L is bigger, the computational efforts become unjustifiable large. The convergence rate is $O(\tau + h^2)$, if we compute the solution with ABCM. For problem (V_2), supp $\Phi'(x) = \infty$ (just as in this case), the algorithm of ABCM implicate two type convolution integrals: concerning V_1^{n-1} and V_N^{n-1}. Due to this terms, which makes the problem nonlocal in time and the interaction of the integrals and different terms, the solution process involves, at any given time step, the history of V_1^{n-1}, V_N^{n-1} and t. For problems (V_1) and (V_2), supp $\Phi'(x) < \infty$, the convolution integrals concern only V_N^{n-1}. Also the summation by S is missing. Thus the CPU time of the computations is approximately two times less, than this, shown in Table 1,2.

Note that the choice of time step: $\tau = h^2$ (Table 1) leads to long time computations and CPU time of ABCM is large. If for example $\tau = h$, the CPU time of ABCM and standard method is $\sim \rho$: $\frac{2^{k+1}}{h} \leq \rho \leq \frac{2^{k+2}}{h}$ ($k=1$ for $h=0.1$, $k=2$ for $h=0.05,\ldots$) and $\sim \frac{1}{h}$ times less, respectively, but the accuracy of ABCM ia still better (in comparison with standard method). For example: CPU time for computations with ABCM, $\tau = h$, $t=0.5$, $L=5$ and $h=0.025$ is 0.719 and the corresponding one with standard method is 0.625 (also for $L=5$).

TABLE 2. Max errors and CPU times

	$l=0.5$	$l=1$	$l=1.5$	$l=2$	
E_∞^h	6.71998e-4	6.75316e-4	6.77910e-4	6.82012e-4	$S=10$
CPU	*0.359*	*0.469*	*0.594*	*0.609*	
E_∞^h	6.70188e-4	6.73815e-4	6.75822e-4	6.79556e-4	$S=20$
CPU	*0.610*	*0.719*	*0.875*	*0.891*	
E_∞^h	6.69881e-4	6.73579e-4	6.75601e-4	6.79273e-4	$S=40$
CPU	*1.031*	*1.093*	*1.266*	*1.390*	
E_∞^h	6.69876e-4	6.73573e-4	6.75592e-4	6.79266e-4	$S=80$
CPU	*2.047*	*2.109*	*2.172*	*2.281*	
E_∞^h	6.69876e-4	6.73573e-4	6.75592e-4	6.79266e-4	$S=1000$
CPU	*19.766*	*19.844*	*20.125*	*20.657*	

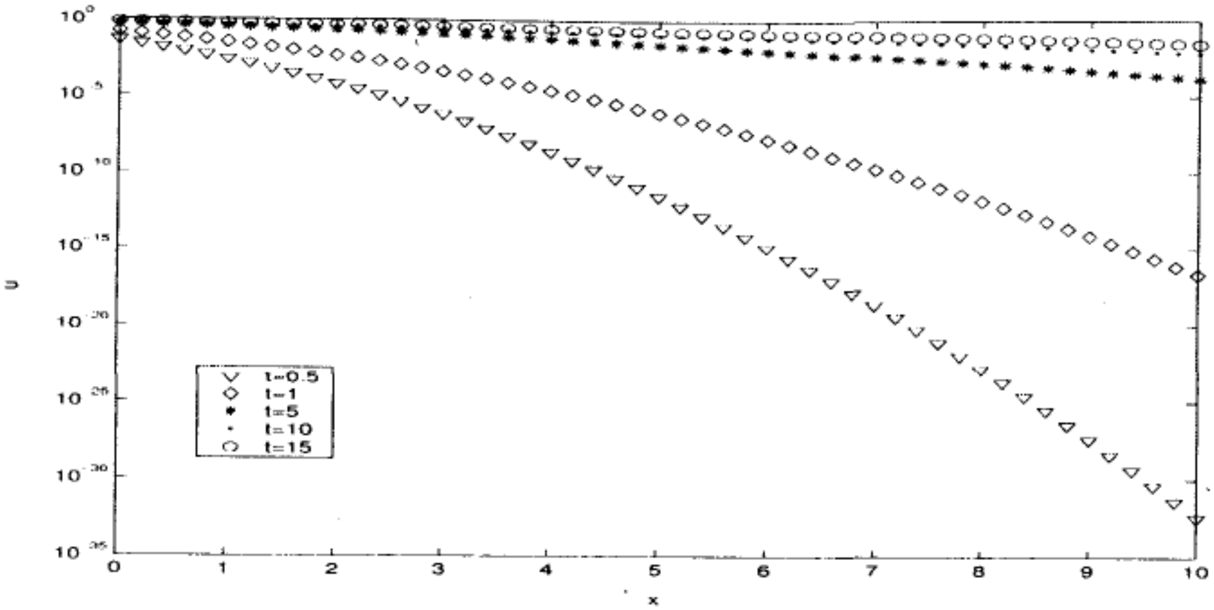

Fig. 1. Exact solution at different time

In Table 2 we give the max errors and CPU times of the numerical solution at $t=0.5$, computed with ABCM for different values of l and S, $\tau=h$, $h=0.025$ and $L=5$.

Example 2. Let $\Phi(x)\equiv 1$, $F_2(u_x(0,t),t)=u_x(0,t)+\frac{1}{\sqrt{\pi t}}e^{-\frac{1}{t}}$, $g(t)=\operatorname{erfc}\frac{1}{\sqrt{t}}$ and $h_2(x)\equiv 0$. Then the exact solution of problem (P_2) is $u(x,t)=\operatorname{erfc}\frac{x+2}{2\sqrt{t}}$. The shape of the solution's profile stretches in x (as $t\to\infty$), see Figure 1. Using the standard method, we take a risk: to compute the solution in large enough (for some t) interval and then it turns out that this interval is not enough large for bigger t. In this situation the ABCM is still effective. On Figure 2, 3, 4 are plotted exact solution and numerical one, obtained by standard method and ABCM for different time levels, $\tau=h=0.025$, $l=0.5$ and $L=5$.

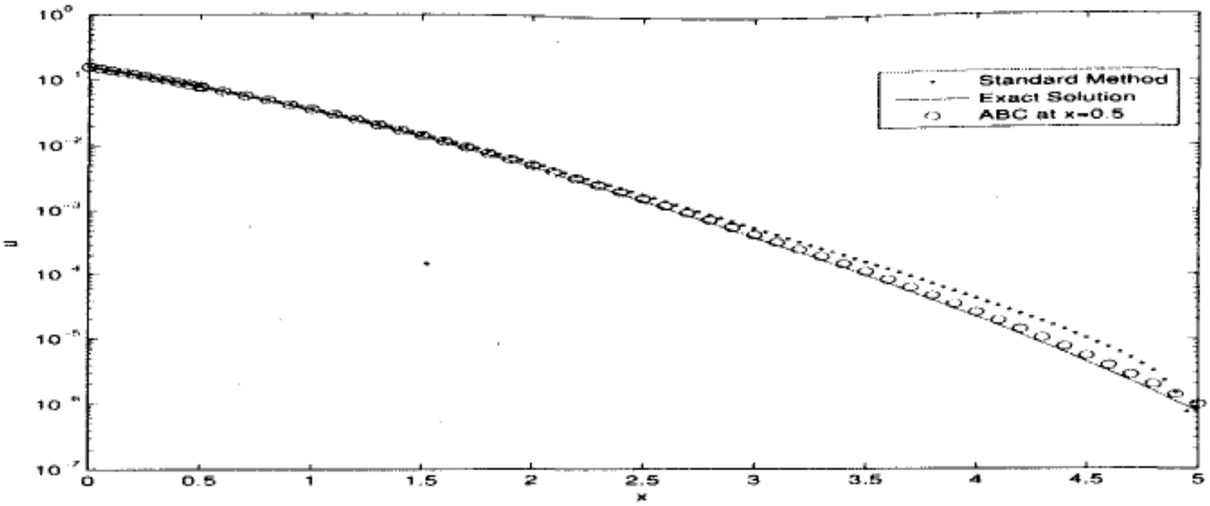

Fig. 2. Numerical and exact solution at $t = 1$

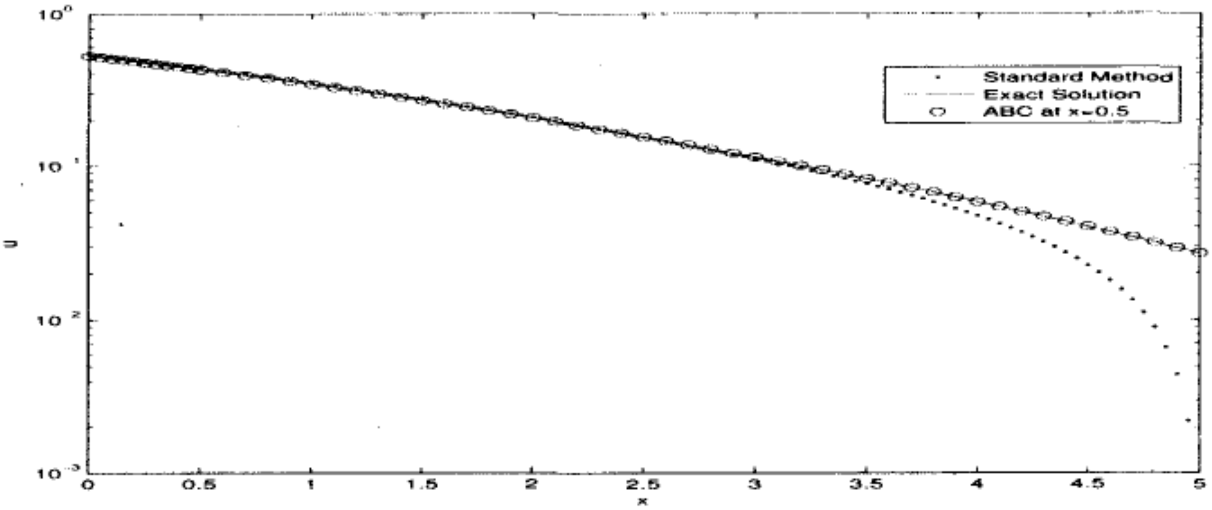

Fig. 3. Numerical and exact solution at $t = 5$

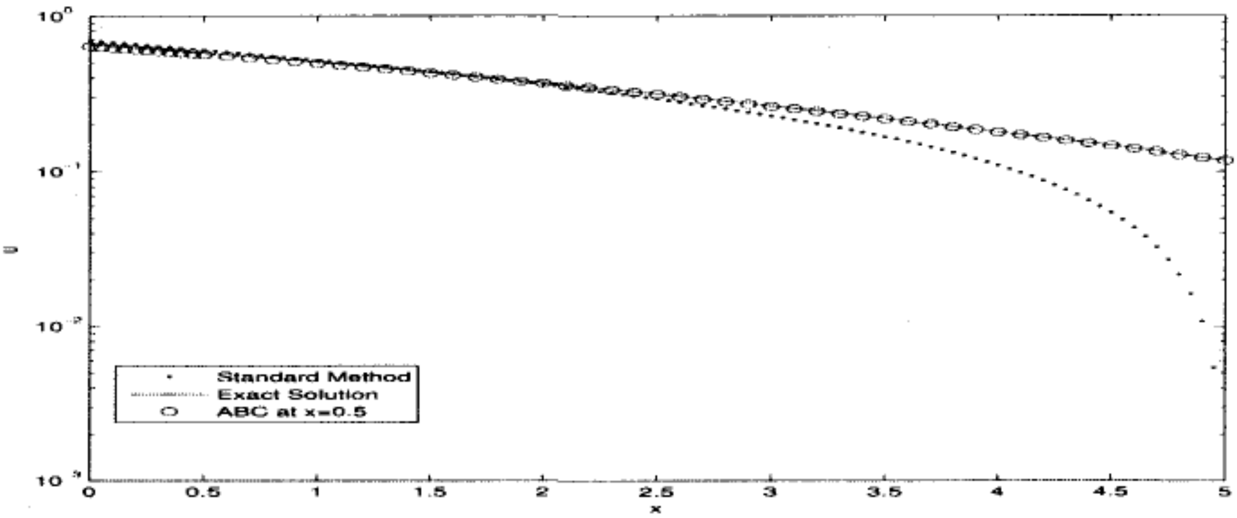

Fig. 4. Numerical and exact solution at $t = 10$

Below we summarize advantages and disadvantages of our numerical algorithms in comparison with known ones: using approximate finite boundary or infinite quasi-uniform grids (uniform or quasi-uniform), etc.

On the positive side are the following features of the schemes:

▷ The high accuracy of the numerical solution.

▷ The scheme on the bounded sub-domain has second order local truncation error in space and first in time. It is not difficult to construct the scheme from high order accuracy as well in time, using three level time scheme, see [1, 16];

▷ The solution in counterpart domains can be computed at any point directly, using formula.

▷ The computations can be performed on a very small region.

▷ The artificial boundary, say l, can be chosen in a very simple way: *supp* $h_i'(x) < l$. Since our boundary condition is exact, the smaller the l, the smaller the computational domain and consequently the less the computational amount.

On the negative side are the following three main disadvantages:

▷ The construction of artificial boundary condition is possible for a restricted class of problems and it's derivation is often not easy;

▷ Geometrically not universal.

▷ Algorithmically simple, but numerically expensive, because of involving the convolution integral with 'memory property', but related only with one point: $x = l$. We could not succeeded to cope with this problem because of the singularity of the integrals kernels. Straightforward evaluation of those convolution requires storing the information along the artificial boundary for all times since $t = 0$ and re-processing this information a each time step. Nevertheless, the performance of the presented schemes (in terms of CPU time or number of operations) is less than that of a standard finite element scheme with no artificial boundary condition, but with sufficiently long domain and zero Dirichlet boundary condition on the remote boundary, such that the the accuracy of both schemes is about the same (the mesh density is one and the same), see [10]. The CPU time of the erf (x) integral is the same as the one of the function $\sin x$ or $\cos x$.

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