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GEOMETRY AND SOLUTIONS OF THE PLANAR PROBLEM OF TWO CENTERS OF GRAVITATION

ASSEN LASHKOV, ANGEL ZHIVKOV

The planar problem of two centers of gravitation was studied by Euler, who found a second "momentum-like" integral and thus the problem turned out to be completely integrable. We present some effective solutions of the motion of the free particle under the influence of the two centers. These solutions are expressed by elliptic theta functions. We also classify all types of such motions from topological point of view. There exist exactly 16 types of motions. Ten of them are unbounded and six are bounded.

Keywords: integrability, general solution, topological classification

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1. INTRODUCTION

One of the famous integrable problems of classical mechanics is the problem of two centers of gravitation, i. e. the problem of determining the motion of a particle in a plane, attached by two fixed centers of force in the plane. Its integrability was discovered by Euler in 1760 [4].

In his "Vorlesungen über Dynamik" [5], Jacobi separated the variables and integrated the equations of motion in terms of elliptic coordinates. Solution in four Jacobi's theta functions is due to Köningsberger [6]. However, both above mentioned solutions contain complete Abelian integrals and that's why they are not convenient for use.

In Theorem 2 below we present effective solutions of the problem of two centers of gravitation. These solutions are expressed in terms of four Jacobi's theta functions and depend on six arbitrary complex constants of motion, including the masses of the centers.

The present article is devoted to the topological classification of the real motions. They depend on six real constants of motion. This problem was studied by Charlier [1] and later by Deprit [2], see also [8, 9]. The case of equal masses of the centers was studied in [3].

In order to define topological invariants of each real solution, it is necessary to define the so–called bifurcation set $\mathbb{B}$. This set $\mathbb{B}$ includes all singular solutions and separates the phase space into connected parts, namely into topologically different types of solutions.

According to our definition, a solution is singular if and only if some Jacobi's theta function[1] degenerates, i. e. became exponent, sinh or cosh. We shall consider a solution $u(t)$ to be topologically equal to the solution $u(-t)$, t being the time. Indeed, the change $t \mapsto -t$ just turns on the opposite the direction on each trajectory.

We also should not make difference between any two solutions, symmetrical according to the line, joining the centers.

The main result of the article is those from Theorem 1: there exist exactly 16 topologically different solutions. Let us remark that any clear definition of topologically equal or different solutions could not be found in the relevant articles, which makes impossible to compare in details our with others' results.

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2. SEPARATION OF VARIABLES

Let $2c$ denote the distance between the two centers. Take the point in the middle of the interval between them as origin and the line joining them as axis of x, so their coordinates will be $(c, 0)$ and $(-c, 0)$.

Denote also by $(x, y) = (x(t), y(t))$ the coordinates of the particle. According Newton's law, the motion of the particle is governed by the equations

$$\begin{aligned} m\,\frac{d^2x}{dt^2} &= -\frac{Gmm_1\,(x-c)}{\left[(x-c)^2+y^2\right]^{\frac{3}{2}}} - \frac{Gmm_2\,(x+c)}{\left[(x+c)^2+y^2\right]^{\frac{3}{2}}}\,, \\ m\,\frac{d^2y}{dt^2} &= -\frac{Gmm_1\,y}{\left[(x-c)^2+y^2\right]^{\frac{3}{2}}} - \frac{Gmm_2\,y}{\left[(x+c)^2+y^2\right]^{\frac{3}{2}}}\,, \end{aligned}$$

where m_1, m_2, m are the masses of the centers and the particle respectively, G is the constant of gravity.

[1] Which takes part on that solution.

By properly choosing the units of distance and mass, we can achieve $c = 1$ and $G = 1$, so that Newton's equations become

$$\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{\mu_1 (x-1)}{\left[(x-1)^2+y^2\right]^{\frac{3}{2}}} - \frac{\mu_2 (x+1)}{\left[(x+1)^2+y^2\right]^{\frac{3}{2}}} ,\\ \frac{d^2y}{dt^2} &= -\frac{\mu_1 y}{\left[(x-1)^2+y^2\right]^{\frac{3}{2}}} - \frac{\mu_2 y}{\left[(x+1)^2+y^2\right]^{\frac{3}{2}}} ,\end{aligned}$$

with x, y, μ_1 and μ_2 dimensionless.

Any ellipse or hyperbola with the two centers as foci is a possible orbit when any of the centers acts alone. It is therefore natural, in defining the position of the particle, to replace the rectangular coordinates (x, y) by elliptic coordinates (p, q):

$$x = p\,q\ , \qquad y = \pm\sqrt{(p^2-1)(1-q^2)}$$

and the inverse

$$\begin{aligned}2p &= \sqrt{(x+1)^2+y^2} + \sqrt{(x-1)^2+y^2}\ ,\\ 2q &= \sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2}\ .\end{aligned}$$

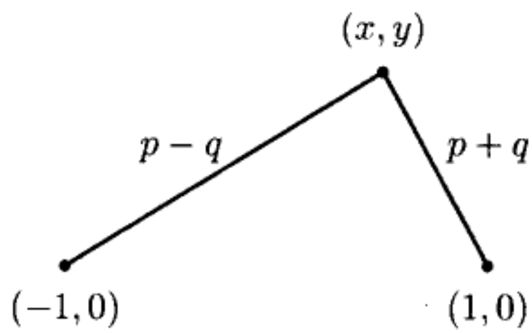

Fig.1. Elliptic coordinates (p, q)

The equations $p = \mathit{constant}$ and $q = \mathit{constant}$ then represent respectively ellipses and hyperbolas whose foci are at the centers $(\pm 1, 0)$. These ellipses and hyperbolas are a particular family of confocal orbits, namely if $m_1 = 0$ or $m_2 = 0$.

In addition to the integral of the total energy

$$\begin{aligned}h &= \frac{\dot{x}^2+\dot{y}^2}{2} - \frac{\mu_1}{\sqrt{(x-1)^2+y^2}} - \frac{\mu_2}{\sqrt{(x+1)^2+y^2}}\\ &= \frac{p^2-q^2}{2}\left(\frac{\dot{p}^2}{p^2-1}+\frac{\dot{q}^2}{1-q^2}\right) - \frac{\mu_1}{p-q} - \frac{\mu_2}{p+q}\\ &= \mathit{constant}\ ,\end{aligned}$$

there also exists an extraordinary "momentum–like" integral [4]

$$\begin{aligned}\gamma &= \frac{(p^2-q^2)^2\,\dot p^2}{2\,(p^2-1)} - h\,p^2 - (\mu_1+\mu_2)\,p \\ &= \frac{(p^2-q^2)^2\,\dot q^2}{2\,(q^2-1)} - h\,q^2 - (-\mu_1+\mu_2)\,q \\ &= \mathit{constant}\,.\end{aligned}$$

For details see [10]. The last equations can be rewritten as

$$\begin{aligned}\tfrac{1}{2}\left(p^2-q^2\right)^2\dot p^2 &= (p^2-1)\left[\,h\,p^2+(\,\mu_1+\mu_2)\,p+\gamma\,\right],\\ \tfrac{1}{2}\left(p^2-q^2\right)^2\dot q^2 &= (q^2-1)\left[\,h\,q^2+(-\mu_1+\mu_2)\,q+\gamma\,\right].\end{aligned}$$

In order to separate finally the variables p and q, we introduce an appropriate scaling of time $s=s(t)$ as follows:

$$dt = \left(p^2-q^2\right)ds \qquad \text{or, equivalently,} \qquad s = \int\limits_0^t \frac{dt}{p^2-q^2}\,.$$

Now the equations of motion

$$\begin{aligned}\frac{1}{2}\left(\frac{dp}{ds}\right)^2 &= (p^2-1)\left[\,h\,p^2+(\,\mu_1+\mu_2)\,p+\gamma\,\right],\\ \frac{1}{2}\left(\frac{dq}{ds}\right)^2 &= (q^2-1)\left[\,h\,q^2+(-\mu_1+\mu_2)\,q+\gamma\,\right].\end{aligned} \tag{2.1}$$

separate into motions of p– and q–variables.

3. TOPOLOGY OF THE SOLUTIONS

We shall discuss the topological types of the solutions of (2.1) in terms of the zeros p_1, p_2, q_1 and q_2 of the polynomials

$$\begin{aligned}L(p) &= h\,p^2+(\,\mu_1+\mu_2)\,p+\gamma \;=\; h\,(p-p_1)(p-p_2)\,,\\ M(q) &= h\,q^2+(-\mu_1+\mu_2)\,q+\gamma \;=\; h\,(q-q_1)(q-q_2)\,,\end{aligned}$$

where h and γ are arbitrary real numbers, μ_1 and μ_2 are real and positive.

Definition 3.1. Two solutions of (2.1),

$$\begin{aligned}p^* &= p^*(\,s\,;\,h^*,\gamma^*,\,\mu_1^*\,,\mu_2^*\,)\\ q^* &= q^*(\,s\,;\,h^*,\gamma^*,\,\mu_1^*\,,\mu_2^*\,)\end{aligned} \qquad \text{and} \qquad \begin{aligned}p^{**} &= p^{**}(\,s\,;\,h^{**},\gamma^{**},\,\mu_1^{**},\mu_2^{**}\,)\\ q^{**} &= q^{**}(\,s\,;\,h^{**},\gamma^{**},\,\mu_1^{**},\mu_2^{**}\,)\end{aligned}\;,$$

are *topologically equivalent* provided there exist some continuous functions

$$h(\lambda)\ ,\ \gamma(\lambda)\ ,\ \mu_1(\lambda)\ ,\ \mu_2(\lambda)\ ,\ p_0(\lambda)\ ,\ q_0(\lambda)\ , \qquad \lambda \in [0,1]$$

which connect respectively h^* and h^{**}, γ^* and γ^{**}, ..., $p^*{}_{|s=0}$ and $p^{**}{}_{|s=0}$, $q^*{}_{|s=0}$ and $q^{**}{}_{|s=0}$. Moreover, the four-tuples $\big(h(\lambda), \gamma(\lambda), \mu_1(\lambda), \mu_2(\lambda)\big)$ should never belong to the bifurcation set

$$\mathbb{B} \;=\; \left\{ \begin{array}{l} \big(h, \gamma, \mu_1, \mu_2\big) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+ \,: \\ \qquad h\,(p_1^2-1)\,(p_2^2-1)\,(p_1-p_2)\,(q_1^2-1)\,(q_2^2-1)\,(q_1-q_2) \;=\; 0 \end{array} \right\}.$$

Equivalently, the roots of the quartics $(p^2-1)L(p)$ and $(q^2-1)M(q)$ remain simple when λ varies from 0 to 1.

By definition, $\mathbb{B}$ consists of all singular solutions of the problem. We consider each possible position of the roots p_1, p_2, q_1, q_2, as well as the initial conditions $p(0)$ and $q(0)$ to prove the main result of the paper.

Theorem 3.1. *There exist exactly 16 topologically different types of solutions in the problem of two centers of gravitation. Six types of solutions are unbounded, another ten types of solutions are bounded :*

Orbits with positive energy ($h > 0$)

1.	$p_1 < -1 < p_2 < 1$	$1 \le p$	$q_1 < -1 < q_2 < 1$	$q \in [-1, q_1]$
2.	$p_1 < -1 < p_2 < 1$	$1 \le p$	$-1 < q_1 < q_2 < 1$	$q \in [q_1, q_2]$
3.	$p_1 < -1 < p_2 < 1$	$1 \le p$	$-1 < q_1 < 1 < q_2$	$q \in [q_2, 1]$
4.	$p_1 < -1 < p_2 < 1$	$1 \le p$	$q_1 < -1 < 1 < q_2$	$q \in [-1, 1]$
5.	$-1 < p_1 < p_2 < 1$	$1 \le p$	$-1 < q_1 < q_2 < 1$	$q \in [q_1, q_2]$
6.	$p_1 < -1 < 1 < p_2$	$p_2 \le p$	$q_1 < -1 < 1 < q_2$	$q \in [-1, 1]$

Orbits with negative energy ($h < 0$)

7.	$-1 < p_1 < 1 < p_2$	$p \in [1, p_2]$	$q_1 < q_2 < -1$	$q \in [-1, 1]$
8.	$-1 < p_1 < 1 < p_2$	$p \in [1, p_2]$	$q_1 < -1 < q_2 < 1$	$q \in [q_2, 1]$
9.	$-1 < p_1 < 1 < p_2$	$p \in [1, p_2]$	$-1 < q_1 < q_2 < 1$	$q \in [-1, q_1]$
10.	$-1 < p_1 < 1 < p_2$	$p \in [1, p_2]$	$-1 < q_1 < q_2 < 1$	$q \in [q_2, 1]$
11.	$-1 < p_1 < 1 < p_2$	$p \in [1, p_2]$	$-1 < q_1 < 1 < q_2$	$q \in [-1, q_1]$
12.	$-1 < p_1 < 1 < p_2$	$p \in [1, p_2]$	$1 < q_1 < q_2$	$q \in [-1, 1]$
13.	$-1 < p_1 < 1 < p_2$	$p \in [1, p_2]$	$q_{1,2} \in \mathbb{C}\setminus\mathbb{R}$	$q \in [-1, 1]$
14.	$1 < p_1 < p_2$	$p \in [p_1, p_2]$	$q_1 < q_2 < -1$	$q \in [-1, 1]$
15.	$1 < p_1 < p_2$	$p \in [p_1, p_2]$	$1 < q_1 < q_2$	$q \in [-1, 1]$
16.	$1 < p_1 < p_2$	$p \in [p_1, p_2]$	$q_{1,2} \in \mathbb{C}\setminus\mathbb{R}$	$q \in [-1, 1]$.

We remind the reader that the Jacobi theta functions are given by their Fourier series

$$
\begin{aligned}
\theta_{00}(z,\tau) &= 1 + 2\,\mathrm{e}^{\pi i\tau}\cos 2\pi z + 2\,\mathrm{e}^{4\pi i\tau}\cos 4\pi z + 2\,\mathrm{e}^{9\pi i\tau}\cos 6\pi z + \cdots\ ,\\
\theta_{01}(z,\tau) &= 1 - 2\,\mathrm{e}^{\pi i\tau}\cos 2\pi z + 2\,\mathrm{e}^{4\pi i\tau}\cos 4\pi z - 2\,\mathrm{e}^{9\pi i\tau}\cos 6\pi z + \cdots\ ,\\
\theta_{10}(z,\tau) &= \quad 2\,\mathrm{e}^{\frac{\pi i\tau}{4}}\cos \pi z + 2\,\mathrm{e}^{\frac{9\pi i\tau}{4}}\cos 3\pi z + 2\,\mathrm{e}^{\frac{25\pi i\tau}{4}}\cos 5\pi z + \cdots\ ,
\end{aligned}
$$

for $z \in \mathbb{C}$ and $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$. The fourth Jacobi's theta will not take part in the solutions. Introduce also the notations

$$
\begin{aligned}
\Theta_{00}(z) &= \theta_{00}(0,\tau)\,\theta_{00}(z,\tau)\ , &\quad \Theta^*_{00}(z^*) &= \theta_{00}(0,\tau^*)\,\theta_{00}(z^*,\tau^*)\ ,\\
\Theta_{01}(z) &= \theta_{01}(0,\tau)\,\theta_{01}(z,\tau)\ , &\quad \Theta^*_{01}(z^*) &= \theta_{01}(0,\tau^*)\,\theta_{01}(z^*,\tau^*)\ ,\\
\Theta_{10}(z) &= \theta_{10}(0,\tau)\,\theta_{10}(z,\tau)\ , &\quad \Theta^*_{10}(z^*) &= \theta_{10}(0,\tau^*)\,\theta_{10}(z^*,\tau^*)\ .
\end{aligned}
$$

Theorem 4.2. *The general solution of the planar problem of two centers of gravitation reads*

$$
x = p\,q = \frac{\Theta_{00}(z) + \operatorname{th}\alpha.\Theta_{10}(z)}{\Theta_{10}(z) + \operatorname{th}\alpha.\Theta_{00}(z)}\cdot\frac{\Theta^*_{00}(z^*) + \operatorname{th}\alpha^*.\Theta_{10}(z^*)}{\Theta^*_{10}(z^*) + \operatorname{th}\alpha^*.\Theta_{00}(z^*)}
$$

$$
\begin{aligned}
y &= \pm\sqrt{(p^2-1)(1-q^2)}\\
&= \frac{i\,\Theta_{01}(z)\Theta^*_{01}(z^*)}{\left[\operatorname{sh}\alpha.\Theta_{00}(z) + \operatorname{ch}\alpha.\Theta_{10}(z)\right]\left[\operatorname{sh}\alpha^*.\Theta^*_{00}(z)^* + \operatorname{ch}\alpha^*.\Theta^*_{10}(z)^*\right]}\,.
\end{aligned}
$$

The elliptic coordinates

$$
p = \frac{\Theta_{00}(z) + \operatorname{th}\alpha.\Theta_{10}(z)}{\Theta_{10}(z) + \operatorname{th}\alpha.\Theta_{00}(z)} \qquad \text{and} \qquad q = \frac{\Theta^*_{00}(z^*) + \operatorname{th}\alpha^*.\Theta_{10}(z^*)}{\Theta^*_{10}(z^*) + \operatorname{th}\alpha^*.\Theta_{00}(z^*)}
$$

satisfy equations (2.1).

The constants which enter in the above formulas depend on the constants of *motion* $\mu_1, \mu_2, h, \gamma, z_0$ *and* z_0^* *as follows :*

$$\kappa^2 = \frac{h-\gamma+\sqrt{(h+\gamma)^2-(\mu_1+\mu_2)^2}}{h-\gamma-\sqrt{(h+\gamma)^2-(\mu_1+\mu_2)^2}}, \quad \kappa^{*2} = \frac{h-\gamma+\sqrt{(h+\gamma)^2-(\mu_1-\mu_2)^2}}{h-\gamma-\sqrt{(h+\gamma)^2-(\mu_1-\mu_2)^2}},$$

$$K = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-\kappa^2 s^2)}}, \qquad K^* = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-\kappa^{*2} s^2)}},$$

$$\tau = \frac{1}{K}\int_1^{\kappa} \frac{ds}{\sqrt{(1-s^2)(1-\kappa^2 s^2)}}, \qquad \tau^* = \frac{1}{K^*}\int_1^{\kappa^*} \frac{ds}{\sqrt{(1-s^2)(1-\kappa^{*2} s^2)}},$$

$$z = z_0 + \frac{2s}{K\sqrt{h-\gamma-\sqrt{(h+\gamma)^2-(\mu_1+\mu_2)^2}}}, \qquad \tanh 2\alpha = -\frac{\mu_1+\mu_2}{h+\gamma},$$

$$z^* = z_0^* + \frac{2s}{K^*\sqrt{h-\gamma-\sqrt{(h+\gamma)^2-(\mu_2-\mu_1)^2}}}, \qquad \tanh 2\alpha^* = \frac{\mu_1-\mu_2}{h+\gamma}.$$

The proof of the theorem above reduces to a straightforward check of certain relations between elliptic theta functions. These relations are in fact well-known, see for example [7].

In general, the solution are complex for $h, \gamma, \mu_{1,2}, z_0, z_0^* \in \mathbb{C}$.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: assenmath@hotmail.com
zhivkov@fmi.uni-sofia.bg