ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 99

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 99

COMPARISON PRINCIPLE FOR LINEAR NON-COOPERATIVE ELLIPTIC SYSTEMS

G. BOYADZHIEV

This paper presents some sufficient conditions for the validity of the comparison principle for the weak solutions of non - cooperative weakly coupled systems of elliptic second-order PDE.

Keywords: Elliptic systems, eigenvalue problem, comparison principle.
2000 MSC: 35J65, 35K60, 35B05, 35R05

In this paper are considered weakly coupled elliptic systems of the form

$$L_M u = 0 \text{ in a bounded domain } \Omega \in R^n \tag{1}$$

where $L_M = L + M$, L is a matrix operator with null off-diagonal elements

$$L = diag.\,(L_1, L_2, ...L_n),$$

$$L_k u_k = -\sum_{i,j=1}^n D_j\left(a_k^{ij}(x) D_i u_k\right) + \sum_{i=1}^n b_k^i(x) D_i u_k + c_k u_k \text{ in } \Omega, \text{ for } k = 1, 2, ...N,$$

and $M = \{m_{ij}(x)\}_{i,j=1}^N$.

Operators L_k are supposed to be uniformly elliptic ones, i.e. there are constants $\lambda, \Lambda > 0$ such that

$$\lambda\,|\xi|^2 \le \sum_{i,j=1}^n a_k^{ij}(x)\xi_i\xi_j \le \Lambda\,|\xi|^2 \tag{2}$$

for every k and any $\xi = (\xi_1, ...\xi_n) \in R^n$.

As for the smoothness of the coefficients $a_k^{ij}(x), b_k^i(x), c_k$ and m_{ij}, we suppose $a_k^{ij}(x), b_k^i(x) \in W^{1,\infty}(\Omega)$, c_k and m_{ij} are continuous in $\overline{\Omega}$.

Hereafter by $f^-(x) = min(f(x), 0)$ and $f^+(x) = max(f(x), 0)$ are denoted the non-negative and, respectively, the non-positive part of the function f. The same convention is valid for matrixes as well. For instance, we denote by M^+ the non-negative part of M, i.e.$M^+ = \{m_{ij}^+(x)\}_{i,j=1}^N$.

This paper concerns the validity of the comparison principle for weakly-coupled elliptic systems. Let us briefly recall the definition of the comparison principle in a weak sense.

The comparison principle holds in a weak sense for the operator L_M if $(L_M u, v) \le 0$ and $u|_{\partial\Omega} \le 0$ imply $(u, v) \le 0$ in Ω for every $v \in W^{1,\infty}(\Omega)$. (3)

As it is well-known, there is no comparison principle for an arbitrary elliptic system (see Theorem 5 below). On the other hand, there are broad classes of elliptic systems, such that the comparison principle holds frue. One of these classes is constructed using condition (4) (see Theorem 1 below):

There is an eigenvalue λ of L_M and its adjoint operator $L*_M$ and the corresponding eigenfunctions $\tilde{w}, w \in \left(W_{loc}^{2,n}(\Omega) \bigcap C_0(\overline{\Omega})\right)^n$ are positive ones. (4)

Note. By adjoint operator we mean $L^*{}_M = L^* + M^t$, $L^* = diag\,(L_1^*, L_2^*, ..., L_n^*)$, and L_k^* are L^2-adjoint operators to L_k.

More precisely, the class is $C^4 = \{L_M$ satisfies (4) and $\lambda > 0\}$, i.e. C^4 contains linear elliptic systems possessing a positive principal eigenvalue with positive corresponding eigenfunction. In C^4 the necessary and sufficient condition for the validity of the comparison principle for systems (Theorem 1 below) is the same as the one for a single equation (See [1]).

Theorem 1. *Assume that (2), (3) and (4) are satisfied. The comparison principle holds for system (1) if and only if $\lambda > 0$.*

Proof. 1. Assume that the comparison principle does not hold for L_M. Let $\underline{u}, \overline{u} \in W^{1,\infty}(\Omega)$ be an arbitrary weak sub- and super-solution of L_M. Then $u = \underline{u} - \overline{u} \in W^{1,\infty}(\Omega)$ is a weak sub-solution of L_M as well, i.e. $(L_M(u), v) \le 0$ in Ω for any $v \in W^{1,\infty}, v > 0$ and $u \equiv 0$ on $\partial\Omega$. Suppose $u^+ \ne 0$. Then

$$0 \ge \left(L_M u^+, w\right) = \left(u^+, L_M^* w\right) = \lambda\left(u^+, w\right) > 0$$

for λ, w defined in (4).

Therefore $u^+ \equiv 0$, i.e for an arbitrary couple sub- and super-solution of L_M we obtain $\underline{u} \le \overline{u}$.

2. Suppose $\lambda < 0$ and $\tilde{w}$ is the corresponding positive eigenfunction of L_M. Then $L_M(\tilde{w}) = \lambda\tilde{w} \leq 0$ but $\tilde{w} > 0$. Therefore there is no comparison principle for (1). □

Unfortunately, the application of this general theorem faces some odds, all about the fact that condition (4) is uneasy to check. First of all, the existence of the principal eigenvalue does not hold for every system (1) (See [9]). The second obstacle is related to the computation of λ even when it exists.

Another broad class, such that the comparison principle holds true, is the class of so-called cooperative elliptic systems, i.e. the systems with $m_{ij}(x) \geq 0$ for $i \neq j$ (See [8]). Most results on the positivity of the classical solutions of linear elliptic systems with non-negative boundary data are obtained for the cooperative systems (See [5,5,12,13,14,16,17,19]). Comparison principle for the diffraction problem for weakly coupled elliptic and parabolic systems is proved in [2].

The spectrum properties of the cooperative L_M are studied as well. A powerful tool in the cooperative case is the theory of the positive operators (See [15]) since the inverse of the cooperative operator L_{M^-} is positive in weak sense. Unfortunately, this approach cannot be applied to the general case $M \neq M^-$ since L_M is not a positive operator at all. Nevertheless in [18] is given a prove for the validity of the comparison principle for non-cooperative systems obtained by small perturbations of cooperative ones.

In [11] are studied existence and local stability of positive solutions of systems with $L_k = -d_k\Delta$, linear cooperative and non-linear competitive part, and Neumann boundary conditions. Theorem 2.4 in [*] is similar to Theorem 2 in the present article for the case $L_k = -d_k\Delta$ and shares the same idea in the proof of adding a big constant.

Let us recall that the comparison principle was proved in [10] for the viscosity sub-and super-solutions of general fully non-linear elliptic systems

$$G^l(x, u^1, ...u^N, Du^l, D^2u^l) = 0, \quad l = 1, ...N$$

(see also the references there). The systems considered in [10] are degenerate elliptic ones and satisfy the same structure-smoothness condition as the one for a single equation. The first main assumption in [10] guarantees the quasi-monotonicity of the system. Quasi-monotonicity in the non-linear case is an equivalent condition to the cooperativeness in the linear one.

The second main assumption in [10] comes from the method of doubling of the variables in the proof.

Note. For linear equations the positiveness and the comparison principle are equivalent. As for the non-linear case, the positiveness of the solutions is an weaker statement than the comparison result for arbitrary sub-and super-solutions; positiveness can hold without comparison and uniqueness of the solutions at all.

This work extends the results obtained for cooperative systems to the non-cooperative ones. The general idea is the separation the cooperative and competitive part of system (1). Then using the appropriate spectral properties of the cooperative part are derived conditions on the general system. In particular we employee the fact that irreducible cooperative system possesses a principal eigenvalue and the corresponding eigenfunction is a positive one, i.e. condition (4) holds. This way we derive some sufficient conditions for validity of the comparison principle for non-cooperative systems as well.

As a preliminary statement we need the following extension of Theorem 1.1.1 [16]:

Theorem 2. *Every cooperative system L_{M^-} has unique principal eigenvalue with positive corresponding eigenfunction.*

Proof. Let us consider the operator $L_c = L_{M^-} + cI$ where c is a real constant and I is the identity matrix in R^n. Then L_c satisfies the conditions of Theorem 1.1.1 [16] if c is large enough, namely

1. L_c is a cooperative one;
2. L_c is a fully coupled;
3. There is a super-solution φ of $L_c\varphi = 0$.

Conditions 1 and 2 above are obviously fulfilled by L_c, since L_{M^-} is a cooperative and a fully coupled one, and L_c inherits this properties from L_{M^-}.

As for the condition 3, we construct the super solution φ using the principal eigenfunctions of the operators $L_k - c_k$. More precisely, $\varphi = (\varphi_1, \varphi_2, , \varphi_n)$, where $(L_k - c_k)\,\varphi_k = \lambda_k\varphi_k$, and $\lambda_k, \varphi_k > 0$ in Ω. Existence of φ_k is a well - known fact.

We claim that φ is a super solution of L_c if c is large enough, i.e. $\varphi \in \left(W^{2,n}_{loc}(\Omega) \bigcap C(\overline{\Omega})\right)^n$ and $\varphi \geq 0$, $L_c\varphi \geq 0$ and φ is not identical to null in Ω.

Since we have chosen φ_k being the principal eigenfunctions of $L_k - c_k$, we have $\varphi_k \in \left(C^2(\Omega) \bigcap C(\overline{\Omega})\right)$ and $\varphi_k > 0$. The last (remaining) condition to prove is $L_c\varphi \geq 0$.

Let

$$A_k = (L_c\varphi)_k = -\sum_{i,j=1}^{n} D_j\left(a_k^{ij}(x) D_i\varphi_k\right) + \sum_{i=1}^{n} b_k^i(x) D_i\varphi_k + \sum_{i=1}^{n} m_{ki}(x)\varphi_i + (c_k + c)\varphi_k =$$

$$= (\lambda_k + c_k + c)\varphi_k + \sum_{i=1}^{n} m_{ki}(x)\varphi_i$$

We claim that $A_k \geq 0$ for every i.

First of all, if we denote by n the the outer unitary normal vector, then

$$\frac{dA_k}{dn}|_{\partial\Omega} = (\lambda_k + c_k + c)\frac{d\varphi_k}{dn} + \sum_{i=1}^{n} m_{ki}(x)\frac{d\varphi_i}{dn}$$

since $\varphi_i|_{\partial\Omega} = 0$. Therefore $\frac{dA_k}{dn}|_{\partial\Omega} < 0$ for $c > c'$ since $\frac{d\varphi_i}{dn} < 0$ on $\partial\Omega$ (See [14], Theorem 7, p.65) and λ_i is independent on c.

Hence there is a neighbourhood $\Omega_\varepsilon = \{x \in \overline{\Omega} : dist(x, \partial\Omega) < \varepsilon\}$ for some $\varepsilon > 0$, such that

$$\frac{dA_k}{dn}|_{\Omega_\varepsilon} < 0$$

.

Since $A_k = 0$ on $\partial\Omega$, then $A_k > 0$ in Ω_ε

The set $\Omega \setminus \Omega_\varepsilon$ is compact, therefore there is $c" > 0$ such that $A_k > 0$ in the compact set $\Omega \setminus \Omega_\varepsilon$ for $c > c"$, since $\varphi_k > 0$ in $\Omega \setminus \Omega_\varepsilon$.

Considering $c > max(c', c")$ we obtain $A_k > 0$ in Ω, therefore φ is indeed a super - solution of L_c.

The rest of the proof follows the proof of Theorem 1.1.1 [16]. □

Theorem 3. *Let (1) be a weakly coupled system with irreducible cooperative part of $L^*_{M^-}$ such that (2) and (3) are satisfied. Then the comparison principle holds for system (1) if*

$$\left(\lambda + \sum_{k=1}^n m^+_{kj}(x)\right) > 0 \text{ for } j = 1...n \text{ and } x \in \Omega, \tag{5}$$

$$\lambda + m^+_{jj}(x) \geq 0 \text{ for } j = 1...n \text{ and } x \in \Omega, \tag{5'}$$

where λ is the principal eigenvalue of the operator L_{M^-}.

Proof. Suppose all conditions of Theorem 3 are satisfied by L_M but the comparison principle does not hold for L_M. Let $\underline{u}, \overline{u} \in W^{1,\infty}(\Omega)$ be an arbitrary weak sub- and super-solution of L_M. Then $u = \underline{u} - \overline{u} \in W^{1,\infty}(\Omega)$ is a weak sub-solution of L_M as well, i.e. $(L_M(u), v) \leq 0$ in Ω for any $v \in W^{1,\infty}, v > 0$ and $u \equiv 0$ on $\partial\Omega$.

Assume $u^+ \neq 0$. Then for any $v > 0$, $v \in W^{1,\infty}(\Omega)$

$$0 \geq (\mathrm{L}_M u^+, v) = (u^+, \mathrm{L}^*_{M^-} v) + (M^+ u^+, v) \tag{6}$$

is satisfied since $L_M(u^+) \leq 0$.

As L_{M^-} is a cooperative operator, such is $(L_{M^-})^* = L^* + (M^-)^t$ as well. According to Theorem 2 above, there is a unique positive eigenfunction $w \in \left(W^{2,n}_{loc}(\Omega) \bigcap C_0(\overline{\Omega})\right)^n$ such that $w > 0$ and $L^*_{M^-} w = \lambda w$ for some $\lambda > 0$.

Then w is a suitable test-function for (6). Inequality (6) reads for $v = w$ as

$$0 \geq \left(u^+, \mathrm{L}^*_{M^-} w\right) + \left(M^+ u^+, w\right) = \left(u^+, \lambda w\right) + \left(M^+ u^+, w\right)$$

or componentwise

$$0 \geq (u^+_k, \lambda w_k) + \left(\sum_{j=1}^n m^+_{kj} u^+_j, w_k\right) \tag{7}$$

for $k = 1, ...n$.

The sum of inequalities (7) is

$$0 \geq \sum_{k=1}^n \left(\left(u^+_k, \tilde{L}^*_k w_k\right) + \left(\sum_{j=1}^n m^+_{kj} u^+_j, w_k\right)\right) =$$

$$= \sum_{k=1}^{n} \left(u_k^+, \lambda w_k\right) + \sum_{k,j=1}^{n} \left(u_j^+, m_{kj}^+ w_k\right) =$$

$$= \sum_{j=1}^{n} \left(u_j^+, \sum_{k=1}^{n} \left(\delta_{jk}\lambda + m_{kj}^+\right) w_k\right) > 0$$

since $u^+ > 0$, $w_k > 0$, (5) and (5').

The above contradiction proves that $u^+ \equiv 0$ and therefore the comparison principle holds for operator L_M. □

Since in [17] are considered only systems with irreducible cooperative part, the ones with reducible L_{M^-} are excluded of the range of Theorem 3. Nevertheless the same idea is applicable to such systems as well, as it is given it Theorem 4.

Theorem 4. *Assume $m_{ij}^- \equiv 0$ for $i \neq j$ and (2), (3) are satisfied. Then the comparison principle holds for system (1) if*

$$\left(\lambda_j + \sum_{k=1}^{n} m_{kj}^+(x)\right) > 0 \text{ for } j = 1...n \text{ and } x \in \Omega, \tag{8}$$

$$\lambda_j + m_{jj}^+(x) \geq 0 \text{ for } j = 1...n \text{ and } x \in \Omega, \tag{9}$$

where λ_j is the principal eigenvalue of the operator L_j.

Proof. Let all conditions of Theorem 4 be satisfied by L_M but the comparison principle does not hold for $\tilde{L}_{M^+}$. Let $\underline{u}, \overline{u} \in W^{1,\infty}(\Omega)$ be an arbitrary weak sub- and super-solution of $\tilde{L}_{M^+}$. Then $u = \underline{u} - \overline{u} \in W^{1,\infty}(\Omega)$ is a weak sub-solution of $\tilde{L}_{M^+}$ as well, i.e. $(\tilde{L}_{M^+}(u), v) \leq 0$ in Ω for any $v \in W^{1,\infty}, v > 0$ and $u \equiv 0$ on $\partial\Omega$.

Suppose that $u^+ \neq 0$. Then for any $v > 0$, $v \in W_2^{1,\infty}(\Omega)$

$$0 \geq \left(\tilde{L}_{M^+}u^+, v\right) = \left(u^+, \tilde{L}^* v\right) + \left(M^+ u^+, v\right) \tag{10}$$

is satisfied since $\tilde{L}_{M^+}u^+ \leq 0$.

According to Theorem 2.1 in [1], there is a positive principal eigenfunction for the operator $\tilde{L}_k^*$, i.e. $\exists \quad w_k(x) \in C^2(\Omega \bigcap R^1)$ such that $\tilde{L}_k^* w_k(x) = \lambda_k w_k(x)$ and $w_k(x) > 0$. Note that w_k are even classical solutions.

Then the vector-function $w(x) = (w_1(x), ..., w_n(x))$, composed of the principal eigenfunctions $w_k(x)$, is suitable as a test-function in (10).

Componentwise, inequality (10) reads for $v = w$ as

$$0 \geq \left(u_k^+, \tilde{L}_k^* w_k\right) + \left(\sum_{j=1}^{n} m_{kj}^+ u_j^+, w_k\right) \tag{11}$$

for $k = 1, ...n$.

The sum of inequalities (11) is

$$0 \geq \sum_{k=1}^{n} \left(\left(u_k^+, \tilde{L}_k^* w_k\right) + \left(\sum_{j=1}^{n} m_{kj}^+ u_j^+, w_k\right)\right) =$$

$$= \sum_{k=1}^{n} \left(u_k^+, \lambda_k w_k\right) + \sum_{k,j=1}^{n} \left(u_j^+, m_{kj}^+ w_k\right) =$$

$$= \sum_{j=1}^{n} \left(u_j^+, \sum_{k=1}^{n} \left(\delta_{jk} \lambda_j + m_{kj}^+ \right) w_k \right) > 0$$

since $u^+ > 0$, $w_k > 0$, (8) and (9).

The above contradiction proves that $u^+ \equiv 0$ and therefore the comparison principle holds for operator L_M^+. □

Condition (9) is useful for construction of contra-example for the non-validity of comparison principle in general.

Theorem 5. *Let (1) be a weakly coupled system with reducible cooperative part L_{M^-} such that (2) and (3) are satisfied. Suppose that (9) is not true, i.e there is some $j \in \{1...n\}$ such that $\left(\lambda_j + m_{jj}^+(x)\right) < 0$ for any $x \in \Omega$, and $m_{jl}^+ = 0$ for $l \neq l$, $l = 1, ...n$. Then the comparison principle does not hold for system (1).*

Note. In Theorem 5 we need violation of the condition (9) in all Ω.

Proof. Let us suppose for simplicity that $j = 1$. We consider vector-function $w(x) = w_1(x), 0, ..., 0$, where $w_1(x)$ is the principal eigenfunction of L_1.

Then for the first component $(L_M)_1$ of L_M is valid

$$(L_M w)_1 = \lambda w_1(x) + m_{11}^+ w_1(x) < 0 \quad \text{in} \quad \Omega$$

where λ_j is the principal eigenvalue of L_1, and $(L_M w)_k = 0$ for $k = 1, ...n$. Therefore, $L_M w \leq 0$ but $w(x) \geq 0$ and comparison principle fails. □

Analogous to Theorem 5 statement is valid for for irreducible systems as well.

Theorem 6. *Let (1) be a weakly coupled system with irreducible cooperative part L_{M^-} such that (2) and (3) are satisfied. Suppose that (5) is not true, i.e there is some $j \in \{1...n\}$ such that $\left(\lambda + m_{jj}^+(x)\right) < 0$ for any $x \in \Omega$, and $m_{jl}^+ = 0$ for $l \neq l$, $l = 1, ...n$. Then the comparison principle does not hold for system (1).*

Note. In Theorem 5 we need violation of the condition (5) in all Ω.

The proof of Theorem 6 follows the proof of Theorem 5 with the obvious corrections.

The sufficient conditions in Theorems 3 and 4 are derived from the spectral properties of the cooperative part of (1) - the operator L_{M^-}, or, in other words, comparing the principal eigenvalue of L_{M^+} with the quantities in M^+. In fact the positive matrix M^+ causes a migration of the principal eigenvalue of L_{M^-} to the left.

Theorems 3 and 4 provide a huge class of non-cooperative systems such that the comparison principle is valid for. The idea of migrating the spectrum of a positive operator on the right works in this case, though the spectrum itself is not studied in this article. The results for non-cooperative systems in this paper are not sharp and the validity of the comparison principle is to be determined more precisely in the future.

REFERENCES

1. Berestycki, H., L. Nirenberg, S. R. S. Varadhan. The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Commun. Pure Appl. Math.*, **47**, 1994, No.1, 47-92.
2. Boyadzhiev, G., N. Kutev. Diffraction problems for quasilinear reaction-diffusion systems. *Nonlinear Analysis*, **55**, 2003, 905-926.
3. Caristi, G., E. Mitidieri. Further results on maximum principle for non-cooperative elliptic systems. *Nonl. Anal. T. M. A.*, **17**, 1991, 547-228.
4. Coosner, C., P. Schaefer. Sign-definite solutions in some linear elliptic systems. *Peoc. Roy. Soc. Edinb.*, Sect. A, **111**, 1989, 347-358.
5. Figueredo, D. di, E. Mitidieri. Maximum principles for cooperative elliptic systems. *C. R. Acad. Sci. Paris*, Ser. I, **310**, 1990, 49-52.
6. Figueiredo, D. di, E. Mitidieri. A maximum principle for an elliptic system and applications to semi-linear problems. *SIAM J.Math.Anal.*, **17**, 1986, 836-849.
7. Gilbarg, D., N. Trudinger. Elliptic partial differential equations of second order. 2nd ed., Springer - Verlag, New York.
8. Hirsch M. Systems of differential equations which are competitive or cooperative I. Limit sets. *SIAM J. Math. Anal.*, **13**, 1982, 167-179.
9. Hess P. On the Eigenvalue Problem for Weakly Coupled Elliptic Systems. *Arch. Ration. Mech. Anal.*, **81**, 1983, 151-159.
10. Ishii, Sh. Koike. Viscosity solutions for monotone systems of second order elliptic PDEs. *Commun. Part. Diff. Eq.*, **16**, 1991, 1095 - 1128.
11. Li Jun Hei, Juan Hua Wu. Existence and Stability of Positive Solutions for an Elliptic Cooperative System. *Acta Math. Sinica*, 2005, **21**, No 5, pp 1113-1130.
12. Lopez-Gomez, J., M. Molina-Meyer. The maximum principle for cooperative weakly coupled elliptic systems and some applications. *Diff. Int. Eq.*, **7**, 1994, 383-398.
13. Mitidieri, E., G. Sweers. Weakly coupled elliptic systems and positivity. *Math.Nachr.*, **173**, 1995, 259-286.
14. Protter, M., H. Weinberger. Maximum Principle in Differential Equations. Prentice Hall, 1976.
15. Reed, M., B.Simon. Methods of modern mathematical Physics, v.IV: Analysis of operators. Academic Press, New York, 1978.
16. Sweers G. Strong positivity in $C(\overline{\Omega})$ for elliptic systems. *Math. Z.*, **209**, 1992, 251-271.
17. Sweers G. Positivity for a strongly coupled elliptic systems by Green function estimates. *J. Geometric Analysis*, **4**, 1994, 121-142.
18. Sweers G. A strong maximum principle for a noncooperative elliptic systems. *SIAM J. Math. Anal.*, **20**, 1989, 367-371.

19. Walter W. The minimum principle for elliptic systems. *Appl. Anal.*, **47**, 1992, 1-6.

Received on September 11, 2006

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bontchev str., bl. 8, 1113 Sofia
BULGARIA
E-mail: georgi_boyadzhiev@yahoo.com