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DEFINABILITY ISSUES IN THE ω-TURING DEGREES

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We show the equivalence of the first order definabilities of the jump of the least element and of the jump operator in the upper semilattice of the ω-Turing degrees.

Keywords: Turing reducibility, ω-Turing degrees, degree structures, definability, jump.

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1. INTRODUCTION

The investigation of the reducibilities between sequences of sets of natural numbers is initiated by Soskov. In the work [8] he introduces the ω-*enumeration* reducibility $\leq_\omega$, which compares the informational content of sequences of sets in a way that generalizes the Selman characterizing theorem for the enumeration reducibility[1]. As a preorder, the reducibility $\leq_\omega$ induces a degree structure – the structure $\mathcal{D}_\omega$ of the ω-enumeration degrees. Again in [8] it is given a definition of a jump operation $'$ over the ω-enumeration degrees. In [9] Soskov and Ganchev continue the studying of the structure $\mathcal{D}_\omega$. They derive that $\mathcal{D}_\omega$ is a proper extension of the structure $\mathcal{D}_e$ of the enumeration degrees whose group $\mathrm{Aut}(\mathcal{D}'_\omega)$ of the jump preserving automorphisms is isomorphic to the automorphism group $\mathrm{Aut}(\mathcal{D}_e)$ of $\mathcal{D}_e$. Recently Ganchev and Sariev show that in $\mathcal{D}_\omega$ the jump operation is first-order definable in the language of the structure order. In this way each automorphism of $\mathcal{D}_\omega$ is jump preserving, so the structures of enumeration and ω-enumeration degrees have isomorphic automorphism groups: $\mathrm{Aut}(\mathcal{D}_e)\cong \mathrm{Aut}(\mathcal{D}_\omega)$.

[1] The Selman Theorem states that $A \leq_e B \iff (\forall X \subseteq \omega)[B \leq_{c.e.} X \to A \leq_{c.e.} X]$

The ω-Turing reducibility $\leq_{T,\omega}$ arises as a 'Turing' analogue of $\leq_\omega$ and just like the ω-enumeration reducibility compares the informational content of the sequences of sets of natural numbers. In this computational framework the informational content of a sequence is uniquely determined by the set of the Turing degrees of the sets that *code* the sequence. We say that a set codes a sequence iff *uniformly* in k, it can compute the k-th element of the considered sequence in its k-th Turing jump:

$$X \subseteq \omega \text{ codes } \{A_k\}_{k<\omega} \iff A_k \leq_T X^{(k)} \text{ uniformly in } k.$$

Having this, we shall say that the sequence $\mathcal{A}$ is ω-Turing reducible to the sequence $\mathcal{B}$ iff each set that codes $\mathcal{B}$ also codes $\mathcal{A}$:

$$\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff (\forall X \subseteq \omega)[X \text{ codes } \mathcal{B} \Rightarrow X \text{ codes } \mathcal{A}].$$

This reducibility is introduced in [6], where its basic properties are explored. The relation $\leq_{T,\omega}$ is a preorder on the set of the sequences of sets of natural numbers and in the standard way induces a degree structure – the upper semi-lattice $\mathcal{D}_{T,\omega}$ of the ω-Turing degrees.

Again in [6] is defined a jump operation on sequences, which induces a corresponding jump operation in the degree structure. Namely the jump $\mathcal{A}'$ of the sequence $\mathcal{A}$ is defined in such a way that:

$$X \text{ codes } \mathcal{A}' \iff (\exists Y)[X \equiv_T Y' \ \&\ Y \text{ codes } \mathcal{A}].$$

How $\mathcal{D}_{T,\omega}$ can be seen as an extension of the structure $\mathcal{D}_T$ of the Turing degrees? By the uniform properties of the Turing jump, it is well known that for all $A, X \subseteq \omega$:

$$A \leq_T X \iff A^{(k)} \leq_T X^{(k)} \text{ uniformly in } k.$$

Thus, the informational content of the set A, described in the Turing universe by the set of the degrees of the sets that decides A, is the same as the content of the sequence $\{A^{(k)}\}_{k<\omega}$ in the context of the ω-Turing reducibility. This observation allows us to define a very natural embedding of the Turing degrees into the ω-Turing:

$$\deg_T(A) \longmapsto \deg_{T,\omega}(\{A^{(k)}\}_{k<\omega}).$$

This embedding preserves the order, the least upper bound operation and even the jump. In this way we may assume the Turing degrees as a proper substructure of $\mathcal{D}_{T,\omega}$. But there are much more strong connections between the both structures. In [6] it is shown that $\mathcal{D}_T$ is definable in $\mathcal{D}_{T,\omega}$ by a first-order formula in the language of the structure order and the jump operation. Also it is proved that the group $\mathrm{Aut}(\mathcal{D}_T)$ of the automorphisms of the Turing degrees is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(\mathcal{D}_{T,\omega})$ of $\mathcal{D}_{T,\omega}$ – namely to the subgroup $\mathrm{Aut}(\mathcal{D}'_{T,\omega})$ of the jump preserving automorphisms of the ω-Turing degrees.

The purpose of this paper is to show that in order to prove that the jump operator is first-order definable in the ω-Turing degrees it is sufficient to prove that the jump $\mathbf{0}_{T,\omega}{}'$ of the least element is definable by a first-order formula in the language of the structure order. We also show that the definability of $\mathbf{0}_{T,\omega}{}'$ implies the definability of $\mathcal{D}_T$ only in the language of the structure order.

2. PRELIMINARIES

2.1. BASIC NOTIONS

We shall denote the set of natural numbers by ω. If not stated otherwise, a, b, $c, \ldots$ shall stand for natural numbers, A, B, $C, \ldots$ for sets of natural numbers, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}, \ldots$ for degrees and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}, \ldots$ for sequences of sets of natural numbers. We shall further follow the following convention: whenever a sequence is denoted by a calligraphic Latin letter, then we shall use the Roman style of the same Latin letter, indexed with a natural number, say k, to denote the k-th element of the sequence (we always start counting from 0). Thus, if not stated otherwise, $\mathcal{A} = \{A_k\}_{k<\omega}$, $\mathcal{B} = \{B_k\}_{k<\omega}$, $\mathcal{C} = \{C_k\}_{k<\omega}$, etc. We shall denote the set of all sequences (*of length* ω) of sets of natural numbers by $\mathcal{S}_\omega$.

As usual $A \oplus B$ shall stand for the set $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$. By A^+ we shall denote the set $A \oplus (\omega \setminus A)$.

We assume that the reader is familiar with the notion of Turing reducibility, $\leq_T$, and with the structure of the Turing degrees $\mathcal{D}_T$ (for a survey of basic results on the Turing degree structure we refer the reader to [2, 3, 4].

The relation $\leq_T$ is a preorder on the powerset 2^ω of the natural numbers and induces a nontrivial equivalence relation $\equiv_T$. The equivalence classes under $\equiv_T$ are called Tuirng degrees. The Tuirng degree which contains the set A is denoted by $\deg_T(A)$. The set of all Turing degrees is denoted by $\mathbf{D}_T$. The Tuirng reducibility between sets induces a partial order $\leq_T$ on $\mathbf{D}_T$ by

$$\deg_T(A) \leq_T \deg_T(B) \iff A \leq_T B.$$

We denote by $\mathcal{D}_T$ the partially ordered set $(\mathbf{D}_T, \leq_T)$. The least element of $\mathcal{D}_T$ is the Turing degree $\mathbf{0}_T$ of $\emptyset$. Also, the degree of $A \oplus B$ is the least upper bound of the degrees of A and B. Therefore $\mathcal{D}_T$ is an upper semi-lattice with least element.

The (Turing) jump A' of $A \subseteq \omega$ is defined as the halting problem for machines with an oracle A,

$$A' = \{e \mid \text{the } e\text{-th Turing machine with oracle } A \text{ halts on input } e\}.$$

The jump operation preserves the Turing reducibility, so we can define $\deg_T(A)' = \deg_T(A')$. Since $A <_T A'$, then we have $\mathbf{a} <_T \mathbf{a}'$ for every Turing degree $\mathbf{a}$. The jump operator is uniform, i.e. there exists a recursive function j such that for every sets A and B, if $A \leq_T B$ via the Turing operator with index e, then $A' \leq_T B'$ via the operator with index $j(e)$.

The ω-Turing reducibility and the corresponding degree structure $\mathcal{D}_{T,\omega}$ are introduced by Sariev and Ganchev in [6]. An equivalent, but more approachable definition in the terms of the uniform Turing reducibility is derived again in the same paper. Here we shall present only on the latter definition. According to it, the sequence $\mathcal{A}$ is ω-Turing reducible to the sequence $\mathcal{B}$, denoted by $\mathcal{A} \leq_{T,\omega} \mathcal{B}$, iff for every $n < \omega$,

$$A_n \leq_T P_n(\mathcal{B}) \text{ uniformly in } n.$$

Here, for each $\mathcal{X} \in \mathcal{S}_\omega$, $\mathcal{P}(\mathcal{X})$ is the so-called *jump sequence* of $\mathcal{X}$ and it is defined as the sequence $\{P_k(\mathcal{X})\}_{k<\omega}$ such that: $P_0(\mathcal{X}) = X_0$ and for each $k < \omega$, $P_{k+1}(\mathcal{X}) = (P_k(\mathcal{X}))' \oplus X_{k+1}$.

Clearly $\leq_{T,\omega}$ is a reflexive and transitive relation, and the relation $\equiv_{T,\omega}$ defined by

$$\mathcal{A} \equiv_{T,\omega} \mathcal{B} \iff \mathcal{A} \leq_{T,\omega} \mathcal{B} \text{ and } \mathcal{B} \leq_{T,\omega} \mathcal{A}$$

is an equivalence relation. The equivalence classes under this relation are called ω-Turing degrees. In particular the equivalence class $\deg_{T,\omega}(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_{T,\omega} \mathcal{B}\}$ is called the ω-Tuirng degree of $\mathcal{A}$. The relation $\leq_{T,\omega}$ defined by

$$\mathbf{a} \leq_{T,\omega} \mathbf{b} \iff \exists \mathcal{A} \in \mathbf{a} \exists \mathcal{B} \in \mathbf{b} (\mathcal{A} \leq_{T,\omega} \mathcal{B})$$

is a partial order on the set of all ω-Turing degrees $\mathbf{D}_{T,\omega}$. By $\mathcal{D}_{T,\omega}$ we shall denote the structure $(\mathbf{D}_{T,\omega}, \leq_{T,\omega})$. The ω-Turing degree $\mathbf{0}_{T,\omega}$ of the sequence $\emptyset_\omega = \{\emptyset\}_{k<\omega}$ is the least element in $\mathcal{D}_{T,\omega}$. Further, the ω-Turing degree of the sequence $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}$ is the least upper bound $\mathbf{a} \vee \mathbf{b}$ of the pair of degrees $\mathbf{a} = \deg_{T,\omega}(\mathcal{A})$ and $\mathbf{b} = \deg_{T,\omega}(\mathcal{B})$. Thus $\mathcal{D}_{T,\omega}$ is an upper semi-lattice with least element.

It is not difficult to notice that each sequence and its jump sequence belong to the same ω-Turing degree, i.e. for all $\mathcal{A} \in \mathcal{S}_\omega$,

$$\mathcal{A} \equiv_{T,\omega} \mathcal{P}(\mathcal{A}). \tag{2.1}$$

In this way, $\mathcal{P}(\mathcal{A})$ is an equivalent to $\mathcal{A}$ sequence, whose members are monotone with respect to $\leq_T$ and each its member decides the halting problems of the previous members.

Given a set $A \subseteq \omega$, denote by $A \uparrow \omega$ the sequence $(A, \emptyset, \emptyset, \ldots, \emptyset, \ldots)$. The definition of $\leq_{T,\omega}$ and the uniformity of the jump operation imply that for all sets of natural numbers A and B,

$$A \uparrow \omega \leq_{T,\omega} B \uparrow \omega \iff A \leq_T B. \tag{2.2}$$

The latter equivalence means that the mapping $\kappa : \mathbf{D}_T \to \mathbf{D}_{T,\omega}$, defined by

$$\kappa(\deg_T(X)) = \deg_{T,\omega}(X \uparrow \omega),$$

is an embedding of $\mathcal{D}_T$ into $\mathcal{D}_{T,\omega}$. Further, the so defined embedding κ preserves the order, the least element and the binary least upper bound operation.

We shall refer to κ as *the natural embedding* of the Turing degrees into the ω-Turing degrees. The range of κ shall be denoted by $\mathbf{D}_1$ and shall be called *the natural copy* of the Turing degrees.

The following theorem makes the connection between the original definition of the ω-Turing reducibility and this one we took here.

Theorem 1 *Let $\mathbf{a} \in \mathbf{D}_{T,\omega}$ be a ω-Turing degree and $\mathbf{C} \subseteq \mathbf{D}_{T,\omega}$ be at most countable set of ω-Turing degrees. Let for each $\mathbf{x} \in \mathbf{C}$, $\mathbf{x} \nleq_{T,\omega} \mathbf{a}$. Then there exists $\mathbf{f} \in \mathbf{D}_1$ such that $\mathbf{a} \leq_{T,\omega} \mathbf{f}$ and for each $\mathbf{x} \in \mathbf{C}$, $\mathbf{x} \nleq_{T,\omega} \mathbf{f}$.*

A full proof[2] of this result can be found in the PhD thesis of the first author, [5].

From the above property easily follows that each ω-Turing degree is uniquely determined by the set of the degrees in $\mathbf{D}_1$, which bound it,

$$\mathbf{a} \leq_{T,\omega} \mathbf{b} \iff (\forall \mathbf{x} \in \mathbf{D}_e)[\mathbf{b} \leq_{T,\omega} \kappa(\mathbf{x}) \rightarrow \mathbf{a} \leq_{T,\omega} \kappa(\mathbf{x})], \tag{2.3}$$

and hence, as one can see, $\mathbf{D}_1$ is an automorphism base of $\mathcal{D}_{T,\omega}$.

2.3. THE JUMP OPERATOR

Following the lines of Sariev and Ganchev [6], the ω-Turing jump $\mathcal{A}'$ of $\mathcal{A} \in \mathcal{S}_\omega$ is defined as the sequence $\mathcal{A}' = (P_1(\mathcal{A}), A_2, A_3, \ldots, A_k, \ldots)$.

Note, that $\mathcal{A}' \equiv_{T,\omega} \{P_{k+1}(\mathcal{A})\}_{k<\omega}$, because for each k, $P_k(\mathcal{A}') = P_{1+k}(\mathcal{A})$.

The jump operator is strictly monotone, i.e. $\mathcal{A} \lneq_{T,\omega} \mathcal{A}'$ and $\mathcal{A} \leq_{T,\omega} \mathcal{B} \Rightarrow \mathcal{A}' \leq_{T,\omega} \mathcal{B}'$. This allows to define a jump operation on the ω-Turing degrees by setting

$$\deg_{T,\omega}(\mathcal{A})' = \deg_{T,\omega}(\mathcal{A}').$$

Clearly for all $\mathbf{a}, \mathbf{b} \in \mathbf{D}_{T,\omega}$, $\mathbf{a} <_{T,\omega} \mathbf{a}'$ and $\mathbf{a} \leq_{T,\omega} \mathbf{b} \Rightarrow \mathbf{a}' \leq_{T,\omega} \mathbf{b}'$.

Also the jump operation on ω-Turing degrees agrees with the jump operation on the Turing degrees, i.e. we have

$$\kappa(\mathbf{x}') = \kappa(\mathbf{x})', \text{ for all } \mathbf{x} \in \mathbf{D}_T.$$

We shall denote by $\mathcal{A}^{(n)}$ the n-the iteration of the jump operator on $\mathcal{A}$. Let us note that

$$\mathcal{A}^{(n)} = (P_n(\mathcal{A}), A_{n+1}, A_{n+2}, \ldots) \equiv_{T,\omega} \{P_{n+k}(\mathcal{A})\}_{k<\omega}. \tag{2.4}$$

[2]here we present only a *sketch* of the proof: the idea is to use a similar result for the ω-enumeration degrees. First note that there is an embedding $\iota : \mathbf{D}_T \rightarrow \mathbf{D}_e$ of the Turing degrees into the enumeration degrees such that $\mathbf{a} \leq_T \mathbf{b} \Leftrightarrow \iota(\mathbf{a}) \leq_e \iota(\mathbf{b})$. Similarly, there is an embedding $\iota_\omega : \mathbf{D}_{T,\omega} \rightarrow \mathbf{D}_\omega$ of the ω-Turing degrees into the ω-enumeration degrees such that $\mathbf{a} \leq_{T,\omega} \mathbf{b} \Leftrightarrow \iota_\omega(\mathbf{a}) \leq_\omega \iota_\omega(\mathbf{b})$. And finally, there is an embedding $\kappa_e : \mathbf{D}_e \rightarrow \mathbf{D}_\omega$ of the enumeration degrees into the ω-enumeration degrees such that $\mathbf{a} \leq_e \mathbf{b} \Leftrightarrow \kappa_e(\mathbf{a}) \leq_\omega \kappa_e(\mathbf{b})$. More precisely these mappings are described, for example, in [6]. The property we use in the proof is that for each $\mathbf{a} \in \mathbf{D}_T$, $\iota_\omega(\kappa(\mathbf{a})) = \kappa_e(\iota(\mathbf{a}))$. The last part of the proof is the counterpart result of Theorem 1 concerning the ω-enumeration degrees. The main difference in it is that the degree $\mathbf{f}$ is not only in $\kappa_e[\mathbf{D}_e]$, but additionally is in $\kappa_e \circ \iota[\mathbf{D}_T]$. The proof of this result can be found in [10].

It is clear that if $\mathcal{A} \in \mathbf{a}$, then $\mathcal{A}^{(n)} \in \mathbf{a}^{(n)}$, where $\mathbf{a}^{(n)}$ denotes the n-th iteration of the jump operation on the degree $\mathbf{a}$.

In [6] it is proved that the range of the jump operator is exactly the upper cone over the first jump ${\mathbf{0}_{T,\omega}}'$ of the least element. Again in the same paper, it is shown even a stronger jump inversion property, which do not posses neither the Turing degrees, nor the enumeration degrees. Namely, for each natural number n if $\mathbf{b}$ is above $\mathbf{a}^{(n)}$, then there is a least ω-Turing degree $\mathbf{x}$ above $\mathbf{a}$ with $\mathbf{x}^{(n)} = \mathbf{b}$. We shall denote this degree by $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$. An explicit representative of $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$ can be given by setting

$$I^n_{\mathcal{A}}(\mathcal{B}) = (A_0, A_1, \ldots, A_{n-1}, B_0, B_1, \ldots, B_k, \ldots), \tag{2.5}$$

where each $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$ are arbitrary.

In the case when $\mathbf{a} = \mathbf{0}_{T,\omega}$ and $n = 1$, for the sake of simplicity, we shall use the notation $\mathbf{I}$ instead of $\mathbf{I}^1_{\mathbf{0}_{T,\omega}}$. Sariev and Ganchev [6] show that the operation $\mathbf{I}$ is monotone,

$$\mathbf{0}_{T,\omega}{}' \leq_{T,\omega} \mathbf{x} \leq_{T,\omega} \mathbf{y} \Rightarrow \mathbf{I}(\mathbf{x}) \leq_{T,\omega} \mathbf{I}(\mathbf{y}).$$

3. THE TURING DEGREES GENERATE $\mathbf{D}_{T,\omega}$

Our goal in this section is to prove that the isomorphic copy $\mathbf{D}_1$ of the Turing degrees under the natural embedding κ generates $\mathbf{D}_{T,\omega}$ under the greatest lower bound operation $\wedge$. More specifically, we will prove that for every ω-Turing degree $\mathbf{a}$ there exist degrees $\mathbf{g}$ and $\mathbf{f}$ from $\mathbf{D}_1$ such that $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$. We begin with the simple observation that each ω-Turing degree is bounded by a degree in $\mathbf{D}_1$.

Lemma 2 *Let* $\mathbf{a} \in \mathbf{D}_{T,\omega}$. *Then there is a degree* $\mathbf{g} \in \mathbf{D}_1$ *such that* $\mathbf{a} \leq_{T,\omega} \mathbf{g}$.

Proof. Recall that $\mathbf{a} \lneq_{T,\omega} \mathbf{a}'$. Then by Theorem 1 applied for $\mathbf{C} = \{\mathbf{a}'\}$, there is $\mathbf{g} \in \mathbf{D}_1$, such that $\mathbf{a} \leq_{T,\omega} \mathbf{g}$, but $\mathbf{a}' \not\leq_{T,\omega} \mathbf{g}$. So $\mathbf{g}$ is a degree from $\mathbf{D}_1$, which bounds $\mathbf{a}$. □

Lemma 3 *Let* $\mathbf{a}, \mathbf{g} \in \mathbf{D}_{T,\omega}$ *and* $\mathbf{a} \leq_{T,\omega} \mathbf{g}$. *Then there is a degree* $\mathbf{f} \in \mathbf{D}_1$ *such that* $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$.

Proof. Let $\mathbf{a} \leq_{T,\omega} \mathbf{g}$. Consider the set $\mathbf{C} = \{\mathbf{x} \in \mathbf{D}_{T,\omega} \mid \mathbf{x} \leq_{T,\omega} \mathbf{g} \,\&\, \mathbf{x} \not\leq_{T,\omega} \mathbf{a}\}$. Clearly $\mathbf{C}$ is countable and, hence, by Theorem 1, there exists a degree $\mathbf{f}$ in $\mathbf{D}_1$ such that $\mathbf{a} \leq_{T,\omega} \mathbf{f}$ and for every $\mathbf{x} \in \mathbf{C}$, $\mathbf{x} \not\leq_{T,\omega} \mathbf{f}$.

Finally, let $\mathbf{b} \leq_{T,\omega} \mathbf{g}, \mathbf{f}$. Then $\mathbf{b} \notin \mathbf{C}$ and so $\mathbf{b} \leq_{T,\omega} \mathbf{a}$. Thus $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$. □

Combining the above lemmas, we have the following.

Theorem 4 *Let* $\mathbf{a} \in \mathbf{D}_{T,\omega}$. *Then there are degrees* $\mathbf{g}, \mathbf{f} \in \mathbf{D}_1$ *such that* $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$.

As a corollary we also have that (the isomorphic copy of) the Turing degrees form an automorphism base for the ω-Turing degrees.

Note that for each degree $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$ with $\mathbf{g}, \mathbf{f} \in \mathbf{D}_1$, the jump $\mathbf{a}'$ can be expressed as the greatest lower bound of two degrees $\mathbf{g}_1$ and $\mathbf{f}_1$ from $\mathbf{D}_1$. The next lemma shows that $\mathbf{g}'$ and $\mathbf{f}'$ are such a pair.

Lemma 5 *Let* $\mathbf{a}, \mathbf{g}$ *and* $\mathbf{f}$ *are ω-Turing degrees such that* $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$. *Then* $\mathbf{a}' = \mathbf{g}' \wedge \mathbf{f}'$.

Proof. Let $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$. Then $\mathbf{a} \leq_{T,\omega} \mathbf{g}, \mathbf{f}$ and by the monotonicity of the jump, $\mathbf{a}' \leq_{T,\omega} \mathbf{g}', \mathbf{f}'$.

Now let $\mathbf{b}$ is a lower bound of $\mathbf{g}'$ and $\mathbf{f}'$. Let $\mathbf{b}_1 = \mathbf{b} \vee \mathbf{0}_{T,\omega}{}'$. Then $\mathbf{b} \leq_{T,\omega} \mathbf{b}_1 \leq_{T,\omega} \mathbf{g}', \mathbf{f}'$ and $\mathbf{0}_{T,\omega}{}' \leq_{T,\omega} \mathbf{b}_1$. Let $\mathbf{c} = \mathbf{I}(\mathbf{b}_1)$. Since the jump inversion operation is monotone, we have that $\mathbf{c} = \mathbf{I}(\mathbf{b}_1) \leq_{T,\omega} \mathbf{I}(\mathbf{g}') \leq_{T,\omega} \mathbf{g}$ and $\mathbf{c} = \mathbf{I}(\mathbf{b}_1) \leq_{T,\omega} \mathbf{I}(\mathbf{f}') \leq_{T,\omega} \mathbf{f}$. But $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$, so $\mathbf{c} \leq_{T,\omega} \mathbf{a}$. Thus $\mathbf{b} \leq_{T,\omega} \mathbf{b}_1 = \mathbf{c}' \leq_{T,\omega} \mathbf{a}'$ by the monotonicity of the jump. □

4. A PROPERTY OF THE LEAST TURING DEGREE

The aim of this section is to provide a characterizing property of the least Turing degree $\mathbf{0}_T$, which shall help us later to find a definition of $\mathbf{D}_1$ in the terms of $\mathbf{0}_{T,\omega}{}'$. We start by showing that $\mathbf{0}_T$ is the only degree $\mathbf{x}$ in $\mathcal{D}_T$, such that for each Turing degree $\mathbf{b}$, if $\mathbf{x} \vee \mathbf{b} \geq_T \mathbf{0}'_T$ then necessary $\mathbf{b} \geq_T \mathbf{0}'_T$. In order to do so, we first need the following notion of *minimal complementation*.

Definition 6 *We shall say that the (Turing) degree* $\mathbf{d} >_T \mathbf{0}_T$ *satisfies the minimal complementation property (MCP) if for every degree* $\mathbf{0}_T <_T \mathbf{a} <_T \mathbf{d}$ *there exists a minimal degree* $\mathbf{m} <_T \mathbf{d}$ *such that* $\mathbf{a} \vee \mathbf{m} = \mathbf{d}$ *(and therefore* $\mathbf{a} \wedge \mathbf{m} = \mathbf{0}_T$*):*

$$MCP(\mathbf{d}) \rightleftharpoons (\forall \mathbf{a} < \mathbf{d})[\mathbf{a} \neq \mathbf{0}_T \to (\exists \mathbf{m})[\mathbf{m} \textit{ is minimal} \ \& \ \mathbf{a} \vee \mathbf{m} = \mathbf{d}]].$$

In [1] Lewis proves that every degree $\mathbf{d} \geq_T \mathbf{0}'_T$ satisfies the minimal complementation property.

From here, one can easily derive that if $\mathbf{x}$ is a nonzero Turing degree, then there is a degree $\mathbf{y}$ such that $\mathbf{x} \vee \mathbf{y} \geq_T \mathbf{0}'_T$, but $\mathbf{y}$ is not above $\mathbf{0}'_T$. Indeed, let $\mathbf{x} \in \mathbf{D}_T$ be a nonzero. Then $\mathbf{x}' \geq_T \mathbf{0}'_T$, and hence MCP($\mathbf{x}'$). Since $\mathbf{0}_T <_T \mathbf{x} <_T \mathbf{x}'$ we have a minimal degree $\mathbf{y} <_T \mathbf{x}'$ such that $\mathbf{x} \vee \mathbf{y} = \mathbf{x}'$. But $\mathbf{y}$ is not above $\mathbf{0}'_T$ because it is minimal. Thus $\mathbf{0}'_T \not\leq_T \mathbf{y}$.

Note also, that the formula: $\varphi(\mathbf{x}) \rightleftharpoons (\forall \mathbf{y})[\mathbf{x} \vee \mathbf{y} \geq_T \mathbf{0}'_T \to \mathbf{y} \geq_T \mathbf{0}'_T]$ is satisfied by the Turing degree $\mathbf{0}_T$ of the recursive sets. Thus, we have proven the following proposition.

Lemma 7 *The least element* $\mathbf{0}_T$ *is the only Turing degree* $\mathbf{x}$ *such that*

$$(\forall \mathbf{y})[\mathbf{x} \vee \mathbf{y} \geq_T \mathbf{0}'_T \to \mathbf{y} \geq_T \mathbf{0}'_T].$$

As an end of this section we move to the structure of the ω-Turing degrees, where we shall investigate the degrees defined by the formula φ. Namely, we shall describe all the ω-Turing degrees $\mathbf{x}$ such that

$$(\forall \mathbf{y})[\mathbf{x} \vee \mathbf{y} \geq_{T,\omega} \mathbf{0}_{T,\omega}{}' \rightarrow \mathbf{y} \geq_{T,\omega} \mathbf{0}_{T,\omega}{}'].$$

First let us consider a sequence $\mathcal{X} = \{X_k\}_{k<\omega}$ such that $\mathcal{D}_{T,\omega} \models \varphi(\deg_{T,\omega}(\mathcal{X}))$. In other words, $\mathcal{X}$ is such that for each sequence $\mathcal{Y} = \{Y_k\}_{k<\omega}$ if $\emptyset_\omega{}' \leq_{T,\omega} \mathcal{X} \oplus \mathcal{Y}$ then $\emptyset_\omega{}' \leq_{T,\omega} \mathcal{Y}$. Noting that for each sequence $\mathcal{A} = \{A_k\}_{k<\omega}$, $\emptyset_\omega{}' \leq_{T,\omega} \mathcal{A}$ is equivalent to $\emptyset' \leq_T A_0$, and then using Lemma 7, we conclude that $X_0 \equiv_T \emptyset$.

Now, let $\mathcal{X} = \{X_k\}_{k<\omega}$ be such that $X_0 \equiv_T \emptyset$ and the sequence $\mathcal{Y} = \{Y_k\}_{k<\omega}$ be such that $\emptyset_\omega{}' \leq_{T,\omega} \mathcal{X} \oplus \mathcal{Y}$. Then we have that $\emptyset' \leq_T X_0 \oplus Y_0 \equiv_T Y_0$, and hence $\emptyset_\omega{}' \leq_{T,\omega} \mathcal{Y}$.

Thus, the degrees in $\mathcal{D}_{T,\omega}$, which satisfy the formula φ, are exactly these that contain a sequences whose zeroth element is the empty set. Further we shall denote the set of all these degrees by $\widetilde{\mathbf{D}_1}$,

$$\widetilde{\mathbf{D}_1} = \{\mathbf{x} \in \mathbf{D}_{T,\omega} \mid (\exists \{A_k\}_{k<\omega} \in \mathbf{x})[\ A_0 = \emptyset\ ]\}.$$

5. DEFINABILITY IN THE ω-TURING DEGREES

In [6] Sariev and Ganchev show the first-order definability of the natural copy $\mathbf{D}_1$ of the Turing degrees in $\mathcal{D}_{T,\omega}$ in the terms of the structure order and the jump operation. In this section we shall improve this result by showing that only in the language of structure order and using $\mathbf{0}_{T,\omega}{}'$ as a parameter, we can define $\mathbf{D}_1$ in $\mathcal{D}_{T,\omega}$. As a consequence, we derive that the definability of $\mathbf{0}_{T,\omega}{}'$ implies this one of the whole jump operator.

Theorem 8 *The following are equivalent:*

1. *the jump operator is first-order definable in $\mathcal{D}_{T,\omega}$;*
2. *the jump $\mathbf{0}_{T,\omega}{}'$ of the least element is first-order definable in $\mathcal{D}_{T,\omega}$;*
3. *the isomorphic copy $\mathbf{D}_1$ of the Turing degrees is first-order definable in $\mathcal{D}_{T,\omega}$*

Proof. (1) $\Rightarrow$ (2): obvious;

(2) $\Rightarrow$ (3): Note that the first-order definability of $\mathbf{0}_{T,\omega}{}'$ implies the first-order definability of the set $\widetilde{\mathbf{D}_1}$, defined in the previous section. But using the set $\widetilde{\mathbf{D}_1}$ a simple definition of $\mathbf{D}_1$ can be derived. Indeed, for each $\mathbf{a} \in \mathbf{D}_{T,\omega}$, denote by $\mu(\mathbf{a})$ the least (ω-Turing) degree $\mathbf{x}$, for which exists degree $\mathbf{y} \in \widetilde{\mathbf{D}_1}$ such that $\mathbf{x} \vee \mathbf{y} = \mathbf{a}$. It is not difficult to see that the operation μ is correctly defined. Moreover, for each $\mathbf{a}$, if $\{A_k\}_{k<\omega} \in \mathbf{a}$ then $\mu(\mathbf{a})$ contains the sequence $(A_0, \emptyset, \ldots, \emptyset, \ldots)$. In order to prove this, first note that

$$(A_0, \emptyset, \ldots, \emptyset, \ldots) \oplus (\emptyset, A_1, A_2, \ldots, A_n, \ldots) \equiv_{T,\omega} \{A_k\}_{k<\omega}.$$

Thus the degree of $(A_0, \emptyset, \dots, \emptyset, \dots)$ is such that there is a degree in $\widetilde{\mathbf{D}_1}$ which cups it to $\mathbf{a}$. Suppose now that $\mathbf{x}$ and $\mathbf{y} \in \widetilde{\mathbf{D}_1}$ are such that $\mathbf{x} \vee \mathbf{y} = \mathbf{a}$. Let us fix sequences $\{X_k\}_{k<\omega} \in \mathbf{x}$ and $\{Y_k\}_{k<\omega} \in \mathbf{y}$ with $Y_0 = \emptyset$. Then $\{X_k\}_{k<\omega} \oplus \{Y_k\}_{k<\omega} \equiv_{T,\omega} \{A_k\}_{k<\omega}$. In particular, $X_0 \oplus Y_0 \equiv_T X_0 \equiv_T A_0$. Therefore,

$$(A_0, \emptyset, \dots, \emptyset, \dots) \leq_{T,\omega} (X_0, \emptyset, \dots, \emptyset, \dots) \leq_{T,\omega} \{X_k\}_{k<\omega}.$$

Hence, the range of μ is exactly the copy $\mathbf{D}_1$ of the Turing degrees under the embedding κ:

$$\mathbf{D}_1 = \{\mu(\mathbf{a}) \mid \mathbf{a} \in \mathbf{D}_{T,\omega}\}.$$

Thus the Turing degrees are first-order definable in the structure $\mathcal{D}_{T,\omega}$ of the ω-Turing degrees.

(3) $\Rightarrow$ (1): By Theorem 4 and Lemma 5, for each ω-Turing degree $\mathbf{a}$ there are ω-Turing degrees $\mathbf{g}, \mathbf{f} \in \mathbf{D}_1$, such that

$$\mathbf{a} = \mathbf{g} \wedge \mathbf{f} \quad \text{and} \quad \mathbf{a}' = \mathbf{g}' \wedge \mathbf{f}', \tag{5.1}$$

and if there is another pair of degrees, whose greatest lower bound exists and is equal to $\mathbf{a}$, then the greatest lower bound of their jumps also exists and is equal exactly to $\mathbf{a}'$.

As we stated in the preliminaries, $\mathbf{D}_1$ is closed under the jump and the ω-Turing jump agrees with the Turing jump. Also, by Shore and Slaman [7], the jump operator is definable in the structure $\mathcal{D}_T$ of the Turing degrees. Hence the restriction of the ω-Turing jump operator over $\mathbf{D}_1$ is definable in the structure $(\mathbf{D}_1, \leq_{T,\omega}, \vee)$. Thus, by (5.1), we conclude that the definability of $\mathbf{D}_1$ implies this of the jump. $\square$

The definability of $\mathbf{0}_{T,\omega}{}'$, alas, still remains an open question.

Question 9 *Is the jump* $\mathbf{0}_{T,\omega}{}'$ *of the least element first-order definable in* $\mathcal{D}_{T,\omega}$*?*

One of the main consequences of the definability of the jump operator will be that each automorphism of $\mathcal{D}_{T,\omega}$ is jump preserving[3], i.e. $\mathrm{Aut}(\mathcal{D}'_{T,\omega})$=$\mathrm{Aut}(\mathcal{D}_{T,\omega})$. This combined with the previously mentioned result by Sariev and Ganchev [6] stating the isomorphicity of the groups of the automorphism of the Turing degrees and of the jump preserving automorphism of the ω-Turing degrees, implies that the groups $\mathrm{Aut}(\mathcal{D}_T)$ and $\mathrm{Aut}(\mathcal{D}_{T,\omega})$ are isomorphic.

[3]a mapping $\pi : \mathbf{D}_{T,\omega} \to \mathbf{D}_{T,\omega}$ is said to be *jump preserving*, if for each degree $\mathbf{a} \in \mathbf{D}_{T,\omega}$, $\pi(\mathbf{a}') = \pi(\mathbf{a})'$.

6. REFERENCES

[1] Lewis, A. E. M.: The minimal complementation property above 0'. *Math. Logic Quart.*, **51**(5), 2005, 470–492. MR2163759 (2006d:03066)

[2] Odifreddi, P. G.: *Classical recursion theory*, vol. I. Studies in logic and the foundations of mathematics (R. A. Soare, A. S. Troelstra, S. Abramsky, S. Artemov, Eds.), vol. 125, Elsevier, 1989.

[3] Odifreddi, P. G.: *Classical recursion theory*, vol. II. Studies in logic and the foundations of mathematics (R. A. Soare, A. S. Troelstra, S. Abramsky, S. Artemov, Eds.), vol. 143, Elsevier, 1999.

[4] Rogers, H. Jr.: *Theory of recursive functions and effective computability.* McGraw-Hill Book Company, New York, 1967.

[5] Sariev, A. C.: *Definability in degree structures.* PhD Thesis, Sofia University, 2015 (in Bulgarian).

[6] Sariev, A. C., Ganchev, H.: The ω-Turing degrees. *Ann. Pure Appl. Logic*, **165**(9), 2014, 1512–1532.

[7] Shore, R. A., Slaman, T. A.: Defining the Turing jump. *Math. Res. Lett.*, **6**(5-6), 1999, 711–722.

[8] Soskov, I. N.: The ω-enumeration degrees. *J. Logic Comput.*, **17**, 2007, 1193–1214.

[9] Soskov, I. N., Ganchev, H.: The jump operator on the ω-enumeration degrees. *Ann. Pure Appl. Logic*, **160**(30), 2009, 289–301.

[10] Soskov, I. N., Kovachev, B.: Uniform regular enumerations. *Math. Structures Comput. Sci.*, **16**(5), 2006, 901–924.

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