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REVIEW OF CONTINUUM MECHANICS AND ITS HISTORY PART II. THE MECHANICS OF THERMOELASTIC MEDIA. PERFECT FLUIDS. LINEARLY VISCOUS FLUIDS

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This paper is the second of a series of two articles reviewing the contributions of continuum mechanics and its history. The review is written for the mathematician who is not a specialist in this field, and aims to give an in-depth overview of the mathematics as well as a historical perspective of this field. The first of the two papers [10], Part I. “Deformation and Stress. Conservation Laws. Constitutive Equations”, starts at the very origins of continuum mechanics and brings the reader up to the 1820’s when Navier publishes the system of the general equations of linear elasticity in 1821. The present paper continues, discussing the consequences of this system, some of its simplifications and approaches for solution. It also gives a perspective of how waves propagate in continuous media. Reviewed are also perfect fluids and linearly viscous fluids. At the end, the paper discusses the conditions for compatibility of the stresses.

Keywords: Mechanics of continuous media, continuum mechanics, hydrodynamics, history of continuum mechanics, elasticity, theory of elasticity.

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1. INTRODUCTION

The first attempt to discuss the motion of a continuous medium in more than one dimension occurs in an isolated passage by D. Bernoulli from 1738 [2], §11, paragraph 4.

We are surrounded by matter in the form of continuous media – deformable solids, liquids and gasses. To study how they move in response to forces, while obeying the natural laws, we need two sets of coordinates. **Material coordinates**, also called **Lagrangian coordinates**, are denoted by (X_1, X_2, X_3) and are the coordinates of the material points of the continuous medium at time $t = 0$. Lagrange introduced them in 1788 in [22], part II, section II. **Spatial coordinates**, also known as **Eulerian coordinates**, are denoted by (x_1, x_2, x_3) and are the coordinates of the points of 3-dimensional space (in which we observe the medium) occupied by the medium at time $t > 0$. Since the material coordinates are the coordinates of the material points at an arbitrary initial time $t = 0$, they can serve for all time as names for the *particles* of the material. The spatial coordinates, on the other hand, we think of as assigned once and for all to a point in the Euclidean space. They are the names of *places*. The motion $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ chronicles the places $\mathbf{x}$ occupied by the particle $\mathbf{X}$ in the course of time. Under external influences – forces and heating – the continuous body deforms. *The goal of Continuum Mechanics is to find the family of transformations*

$$x_i = x_i(X_1, X_2, X_3, t) \ , \quad i = 1, 2, 3, \tag{1}$$

giving the Eulerian coordinates as functions of the Lagrangian coordinates for $t \geq 0$. This motion is deterministic, obeying only the natural laws.

The general theory of the motion of a continuous medium, understood as a family of deformations continuously varying in time, is almost exclusively due to Euler, published in the period 1745 – 1766, references [25] – [40] in the first paper of this review, and Cauchy, published in the period 1815 – 1841, references [3] – [18] in the first paper of this review. Important special results were added by D'Alembert in 1749, Green in 1839, Stokes in 1845, Helmholtz in 1858 and Cesaro in 1906, also cited in the first part of this review.

2. LINEAR THERMOELASTIC CONTINUOUS MEDIA

The systems of equations

$$c_{ijkl}\, u_{k,jl} - \chi_{ij}\, T_{,j} + \rho\, f_i = \rho\, \ddot{u}_i \ , \quad i = 1, 2, 3 \qquad \text{equations of motion} \tag{2}$$

$$k_{ij}\, T_{,ij} - c_\varepsilon \frac{\partial T}{\partial t} - \chi_{ij}\, T_0 \frac{\partial u_{i,j}}{\partial t} + \rho\, r = 0 \quad \text{equation of thermal condactivity} \tag{3}$$

for the unknown functions u_i, T, which are valid for any linear **thermoelastic** anisotropic medium, were first published in 1821 by Navier [30]. Here and in the rest of the paper a dot above a variable denotes a differentiation with respect to time and two dots denote a double differentiation with respect to time. Here u_i are the three components of the vector of displacement $\mathbf{u}$, T is the temperature difference, and are functions of the space coordinates (x_1, x_2, x_3) and the time t. As usually in the literature, a first partial derivative with respect to a space coordinate is denoted by one lower index after a comma, a second partial derivative with respect to space coordinates is denoted by two lower indexes after a comma; ρ is the mass density, f_i are the components of the assigned (mass) force, r is the heat source, and c_{ijkl}, χ_{ij}, k_{ij}, c_ε, T_0 are constants.

The system (2), (3) in the case of an isotropic body acquires the form:

$$(\lambda + \mu)\, u_{j,ji} + \mu\, u_{i,jj} - \chi\, T_{,i} + \rho\, f_i = \rho\, \ddot{u}_i \tag{4}$$

$$k\, T_{,ii} - c_\varepsilon \frac{\partial T}{\partial t} - \chi\, T_0 \frac{\partial u_{i,i}}{\partial t} + \rho\, r = 0. \tag{5}$$

Both these systems simplify significantly if the process is isothermal or adiabatic. A process is called **isothermal** if the changes that are taking place are "slow", so that the change in temperature is small and can be ignored. In the notation we use, $T = 0$. In that case we do not consider at all the equation of thermal conductivity. So the remaining equations are

$$c_{ijkl}\, u_{k,jl} + \rho\, f_i = \rho\, \ddot{u}_i\ , \quad i = 1, 2, 3 \tag{6}$$

for an anisotropic body and

$$(\lambda + \mu)\, u_{j,ji} + \mu\, u_{i,jj} + \rho\, f_i = \rho\, \ddot{u}_i \tag{7}$$

for an isotropic medium. The equations (6) and (7) are known as **the isothermal equations of elasticity** for an anisotropic and isotropic medium, respectively. The constants c_{ijkl}, λ, μ are called isothermal constants.

The process is called adiabatic if the changes that take place in the medium are "fast", so that the heat exchange that takes place between different parts of the body, being a "slower" process, can be ignored, that is, $q_i = 0$. Of course, in this case there are no sources of heat.

It is interesting that an adiabatic process is also isoentropic, that is, has a constant entropy $\eta = \eta_0 = constant$. This can be seen from the equations

$$\rho\, T_0 \frac{\partial \eta}{\partial t} + q_{i,i} = \rho\, r \qquad \text{law of conservation of energy}$$

$$\rho\, \eta = \rho\, \eta_0 + \frac{c_\varepsilon}{T_0} T + \chi_{ij}\, \varepsilon_{ij} \qquad \text{constitutive equation for the entropy}$$

which were derived in Part I of this review, as a part of the system of 20 equations which any linear thermoelastic continuous medium obeys. From the constitutive equation for the entropy, for an anisotropic body, we get

$$T = -\frac{\chi_{ij}\, T_0}{c_\varepsilon}\, \varepsilon_{ij}.$$

For an isotropic body the relationship between the entropy and the deformations is

$$T = -\frac{\chi\, T_0}{c_\varepsilon}\, \varepsilon_{ii}.$$

After substituting these last two equations for T in the equations of motion (2) for an elastic anisotropic medium and in equations (4) for a thermoelastic isotropic medium, these equations aquire the form:

$$c^a_{ijkl}\, u_{k,jl} + \rho\, f_i = \rho\, \ddot{u}_i \;, \quad i = 1, 2, 3 \tag{8}$$

and respectively

$$(\lambda^a + \mu^a)\, u_{j,ji} + \mu^a\, u_{i,jj} + \rho\, f_i = \rho\, \ddot{u}_i, \tag{9}$$

where the adiabatic constants c^a_{ijkl}, λ^a and μ^a are related to their corresponding isothermal constants via the equations

$$c^a_{ijkl} = c_{ijkl} + \frac{\chi_{ij}\, \chi_{kl}\, T_0}{c_\varepsilon}$$

$$\lambda^a = \lambda + \frac{\chi^2\, T_0}{c_\varepsilon}, \qquad \mu^a = \mu.$$

In the remaining of the paper we will drop the upper index of the constants in the adiabatic equations of elasticity, namely in equations (8) and equations (9), so they will not differ in form from their corresponding isothermal equations (6) and (7). We will call these equations **the equations of elasticity**.

For an isothermal process, the constitutive equations

$$\sigma_{ij} = c_{ijkl}\, \varepsilon_{kl} - \chi_{ij}\, T \tag{10}$$

for the components of the stress tensor, acquire the form

$$\sigma_{ij} = c_{ijkl}\, \varepsilon_{kl}. \tag{11}$$

In the case of adiabatic process the relationship among stresses and deformations is analogous, if the constants c_{ijkl} are the adiabatic constants.

For isotropic bodies from

$$c_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk})$$

follows that

$$\sigma_{ij} = \lambda\, \varepsilon_{kk}\delta_{ij} + 2\mu\, \varepsilon_{ij}. \tag{12}$$

The equations (11) or respectively (12), giving the relationship between the stresses and the deformations, are known as **the generalized law of Hooke**. Equations (12) with $\lambda = \mu$ were derived from a molecular model by Navier published in 1821 [28], [29]; more generally by Poisson [32] in 1829 .

The **elasticities** λ, μ and c_{ijkl} in equations (11) and (12) are material constants or functions of the temperature or entropy. Their physical dimensions are those of stress, and they bear no physical connection with the mathematically analogous viscosities appearing in the Navier-Poisson law, discussed in section 8 “Linearly Viscous Fluids” of this paper.

The equations (11) or respectively (12) together with the equations of motion

$$\sigma_{ij,j} + \rho\, f_i = \rho\, \ddot{u}_i$$

of a linear continuous medium and the equations of strain

$$\varepsilon_{ij} = \frac{1}{2}\Big(u_{i,j} + u_{j,i}\Big)$$

form **the system of equations of elasticity** for an anisotropic and isotropic body respectively. This system consists of 15 equations for the 15 unknown functions u_i, ε_{ij} and σ_{ij}, where ε_{ij} are the components of the strain tensor and σ_{ij} are the components of the stress tensor. Unlike that, the systems (6) and (7) are systems with 3 equations each for the three unknown displacements u_i. These equations are typically called **the equations of elasticity in displacements** or **equations of Lamè**, and can be solved with appropriate initial and boundary conditions.

Because of the symmetries $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$, the number of the independent components of the tensor c_{ijkl} is significantly smaller than that of a general tensor of rank 4. Thus, it is appropriate to replace couples of indexes with a single index via the following scheme: $11 \to 1,\ 22 \to 2,\ 33 \to 3,$ 23 and $32 \to 4,\ 31$ and $13 \to 5,\ 12$ and $21 \to 6$. The following notation is also used to denote the components of the stress tensor and those of the tensor of deformations:

$$\sigma_1 = \sigma_{11},\ \sigma_2 = \sigma_{22},\ \sigma_3 = \sigma_{33},\ \sigma_4 = \sigma_{23},\ \sigma_5 = \sigma_{31},\ \sigma_6 = \sigma_{12}$$

$$\varepsilon_1 = \varepsilon_{11},\ \varepsilon_2 = \varepsilon_{22},\ \varepsilon_3 = \varepsilon_{33},\ \varepsilon_4 = 2\varepsilon_{23},\ \varepsilon_5 = 2\varepsilon_{31},\ \varepsilon_6 = 2\varepsilon_{12}.$$

Then the generalized law of Hooke (11) acquires the form

$$\sigma_\alpha = c_{\alpha\beta}\, \varepsilon_\beta, \tag{13}$$

where the Greek indices run from 1 to 6, and repeated indices denote summation from 1 to 6. Because $c_{\alpha\beta} = c_{\beta\alpha}$, the number of independent constants in the generalized law of Hooke (13) for an arbitrary anisotropic body is 21.

The function

$$\mathcal{U} = \frac{1}{2}\sigma_{ij}\, \varepsilon_{ij} = \frac{1}{2}\sigma_\alpha\, \varepsilon_\alpha = \frac{1}{2}c_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl} = \frac{1}{2}c_{\alpha\beta}\, \varepsilon_\alpha\, \varepsilon_\beta \tag{14}$$

is called the density of the potential energy of the deformation, or the **elastic potential**. So the potential energy of the deformation is

$$U = \frac{1}{2}c_{ijkl}\int_V \varepsilon_{ij}\varepsilon_{kl} dV = \int_V \mathcal{U}\, dV$$

and

$$\sigma_\alpha = \frac{\partial \mathcal{U}}{\partial \varepsilon_\alpha}.$$

There is a physical reason to require that the elastic potential be a positive definite form, because then, in any given small strain from an unstressed state, the stress must do positive work. Assuming that the elastic potential is positive definite, it follows that the constants $c_{\alpha\beta}$ satisfy the following restrictions: $c_{11} > 0, \dots, \det|c_{\alpha\beta}| > 0$. In the case of isotropic body these inequalities acquire the form $\lambda + 2\mu > 0,\ 4\mu(\lambda+\mu) > 0, \dots, 4\mu^5(3\lambda+2\mu) > 0$. Hence the necessary and sufficient condition for these inequalities to be satisfied is:

$$3\lambda + 2\mu > 0, \quad \mu > 0. \tag{15}$$

The elastic potential and its resulting potential energy of the deformation are due to Green, who published them in 1839 [11], and in 1841 [12]. He proposed that the work done by stress in a deformation depends only upon the strain and is recoverable work. In his original papers, Green defines the **stored energy** Σ by

$$\Sigma(\varepsilon) = \frac{1}{2}\sigma_{km}\varepsilon_{km},$$

(later renamed the elastic potential U, which we defined with (14)). Thus, in Green's theory the number of independent elasticities is 21. He derives that

$$\sigma_{km} = \frac{\partial \Sigma}{\partial \varepsilon_{km}}. \tag{16}$$

By the representation theorem for isotropic scalar functions, it follows that the stored energy can be expressed in terms of the first and second invariants of the tensor ε as

$$\Sigma = \frac{1}{2}(\lambda + 2\mu) I_\varepsilon^2 - 2\,\mu\, II_\varepsilon.$$

A body is called **hyperelastic** if it obeys Green's theory, based upon the use of Σ as a stress potential according to (16). This theory has some remarkable results, which we review next.

The fact that $\det|c_{\alpha\beta}| > 0$ guarantees that the equations of the generalized law of Hooke (13) can be solved for the deformations, obtaining

$$\varepsilon_\alpha = s_{\alpha\beta}\,\sigma_\beta,$$

where the matrix $|s_{\alpha\beta}|$ is the inverse of the matrix of elastic constants $|c_{\alpha\beta}|$, and is called **the matrix of stiffnesses**. In the isotropic case the deformations ε_{ij} can be expressed with the stresses, if we take in consideration that for $i = j$ from the generalized Hooke's law (12), namely, $\sigma_{ij} = \lambda\,\varepsilon_{kk}\delta_{ij} + 2\mu\,\varepsilon_{ij}$, we obtain

$$\sigma_{ii} = (3\lambda + 2\mu)\varepsilon_{ii}.$$

Then

$$\begin{aligned} \varepsilon_{ii} &= \frac{1}{2\mu}(\sigma_{ij} - \lambda\varepsilon_{kk}\delta_{ij}) \\ &= \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{kk}\delta_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}, \end{aligned} \tag{17}$$

where the constant

$$E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}$$

is always positive and is called the module of Jung. For metals the module of Jung is of the order of $10^{11}\,\mathrm{N/m^2}$. The constant ν is called the coefficient of Poisson and is $\nu = \lambda/(2\lambda + 2\mu)$. It is clear that $-1 < \nu < 1/2$. For all known materials Poisson's coefficient is positive. For metals it varies usually in the interval $[1/4,\ 1/3]$.

Let us now consider a couple of special cases. Let us assume that $f_i = 0$ and that the problem is static, i.e., the components u_i of the displacement do not depend on the time t. In this case the initial conditions of the system of differential equations are no longer present and only the boundary conditions $u_i(\mathbf{x}, t) = g_i(\mathbf{x}, t)$ for $x \in S_u$ and $\sigma_{ij}(\mathbf{x}, t)n_j(\mathbf{x}, t) = h_i(\mathbf{x}, t)$ for $x \in S_\sigma$ remain, because the medium is elastic and not thermoelastic.

1. Simple Shear

Simple shear is characterized by the following stresses:

$$\sigma_{23} = \text{constant} \neq 0, \quad \text{the rest of} \quad \sigma_{ij} = 0. \tag{18}$$

These stresses satisfy the equations of equilibrium $\sigma_{ij,j} = 0$. The deformations that correspond to them are:

$$\varepsilon_{23} = \frac{1}{2\mu}\sigma_{23}, \quad \text{the rest of} \quad \varepsilon_{ij} = 0.$$

The geometric interpretation of the tensor of deformations, which was explained in the first part of this review, follows that a cube with sides parallel to the coordinate planes, will deform under a simple shear in such a way that the right angle between the edges of the cube, that are parallel to the axes x_2 and x_3 decreases (if $\sigma_{23} > 0$) or increases (if $\sigma_{23} < 0$) with the angle $\gamma_{23} = 2\varepsilon_{23}$. From equations (10) follows that

$$\mu = \frac{\sigma_{23}}{\gamma_{23}}.$$

Thus, μ has the meaning of the ratio between the so called shearing stress σ_{23} to the resulting from it change $\gamma_{23}$of the right angle. The constant μ is called **module of shearing**, it is often denoted in the technical literature by G.

2. Hydrostatic Pressure

We consider an elastic body with an arbitrary shape. Its boundary is subjected to stresses, that are applied perpendicularly to the surface, toward the body, and have a constant intensity $p > 0$. Then

$$\sigma_i = -p\,n_i, \tag{19}$$

where n_i are the components of the outward unit normal to the surface of the body. The stresses

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \quad \text{and} \quad \sigma_{ij} = 0 \text{ if } i \neq j$$

satisfy the equations of equilibrium with $f_i = 0$ and boundary conditions given by (19). From equations (17) it follows that

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = -\frac{p}{3\lambda + 2\mu}, \quad \text{and} \quad \varepsilon_{ij} = 0 \quad \text{if } i \neq j. \tag{20}$$

Both in this and in the previously considered example, the deformations are constant, and thus satisfy the conditions for compatibility of St. Venant, discussed in detail in the first paper of this review. Hence from them the displacements u_i can be calculated, that correspond to the stresses in consideration. From equations (12) one calculates the relative change in the volume (expansion if $p < 0$) and (contraction if $p > 0$). Let $\varepsilon \equiv \varepsilon_{ii}$. Then

$$\varepsilon = -\frac{p}{k}, \tag{21}$$

where

$$k = \lambda + 2\mu/3 = E/(3 - 6\nu) \tag{22}$$

is called the **module of contraction**. It is the ratio of the hydrostatic pressure to the relative change of volume. From the inequalities (21) it follows that $k > 0$.

The elastic material is called noncompressible if under pressure the relative change ε of the volume remains zero. In that case from (21) and (22) we calculate that $\nu = 1/2$.

3. THE LAW OF CONSERVATION OF MECHANICAL ENERGY

We considered the law of conservation of mechanical energy in part I of this review and showed that it has the form

$$\frac{dK}{dt} + \int_V \sigma_{ij}\, d_{ij}\, dV = W,$$

where $K = \int_V \rho\, v_i\, v_i/2\, dV$ is the kinetic energy,

$$W = \int_V \rho\, f_i\, v_i\, dV + \int_S \sigma_i\, v_i\, dS \tag{23}$$

is the power of the external forces and $d_{ij} \equiv (v_{i,j} + v_{j,i})/2 = d_{ji}$ is the tensor of the rate of deformations, introduced by Euler in 1769 [9], §§ 9-12. In the case of small deformations $d_{ij} = \partial\varepsilon_{ij}/\partial t$. The total time-derivative of the potential energy U of the deformations is

$$\frac{dU}{dt} = \frac{1}{2} c_{ijkl} \int_V \frac{\partial}{\partial t}(\varepsilon_{ij}\varepsilon_{kl}) dV = c_{ijkl} \int_V \varepsilon_{ij} \frac{\partial}{\partial t}\varepsilon_{kl} dV.$$

Then it follows that in the linear theory of elasticity the law of conservation of mechanical energy acquires the form

$$\frac{d}{dt}(K + U) = W. \tag{24}$$

If we denote by

$$A(\tau) = \int_0^\tau W dt \tag{25}$$

the work done by the external forces during the interval of time $[0, \tau]$ and assume that at $t = 0$ the body was in an undeformed state and at rest, i.e. $K(0) = U(0) = 0$, then from equation (24) it follows that

$$K(\tau) + U(\tau) = A(\tau), \tag{26}$$

where the argument of the functions K, U and A determines the moment at which they are evaluated. Equation (26) shows that the sum of the kinetic and the potential energies at a given moment equals the work done by the mass forces and the surface forces upto that moment.

4. THE STATIC PROBLEM

In a static problem we are not interested in the process of deformation, but only in the final state, which we regard as an equilibrium. The static theory is a linear one: uniformly doubled displacements always result from uniformly doubled loads, and, more generally, from displacements $\mathbf{u}^1$, $\mathbf{u}^2$ corresponding to stresses σ^1, σ^2, assigned forces $\mathbf{f}^1$, $\mathbf{f}^2$, and assigned surface loads σ_N^1, σ_N^2 we construct a displacement $\mathbf{u} \equiv \mathbf{u}^1 - \mathbf{u}^2$ answering to the stress $\sigma = \sigma^1 - \sigma^2$, force $\mathbf{f} = \mathbf{f}^1 - \mathbf{f}^2$, and surface load $\sigma_N = \sigma_N^1 - \sigma_N^2$.

Let us assume that such an equilibrium state is reached in the moment $t = \tau$. Then $K(\tau) = 0$ and hence $U(\tau) = A(\tau)$. Since the potential energy $U(\tau)$ does not depend on the "path" of the deformation, but only on the final deformation, we may choose an arbitrary "path" of deformation. Let us choose the mass force components f_i, the stress components σ_i and the components u_i of the displacement in the following way:

In the time interval $0 \le t \le \varepsilon$: $f_i(\mathbf{x}, t) = 0$, $\sigma_i(\mathbf{x}, t) = 0$ and $u_i(\mathbf{x}, t) = 0$,

in the interval $\varepsilon \le t \le \tau - \varepsilon$:

$$f_i(\mathbf{x}, t) = f_i \frac{t - \varepsilon}{\tau - 2\varepsilon}, \quad \sigma_i(\mathbf{x}, t) = \sigma_i \frac{t - \varepsilon}{\tau - 2\varepsilon}, \quad \text{and} \quad u_i(\mathbf{x}, t) = u_i \frac{t - \varepsilon}{\tau - 2\varepsilon},$$

in the interval $\tau - \varepsilon \le t \le \tau$: $f_i(\mathbf{x}, t) = f_i$, $\sigma_i(\mathbf{x}, t) = \sigma_i$ and $u_i(\mathbf{x}, t) = u_i$,

where by f_i, σ_i and u_i we denote the values of these functions at the moment $t = \tau$ and depend only on the position $\mathbf{x}$. They satisfy the equations of equilibrium and so the functions $f_i(\mathbf{x}, t)$, $\sigma_i(\mathbf{x}, t)$ and $u_i(\mathbf{x}, t)$, defined above satisfy the equations of motion. Then from equations (25) and (23) we obtain

$$U(\tau) = \int_0^{\varepsilon} W dt + \int_{\varepsilon}^{\tau-\varepsilon} W dt + \int_{\tau-\varepsilon}^{\tau} W dt = \int_{\varepsilon}^{\tau-\varepsilon} W dt = \frac{1}{2} \int_V \rho f_i u_i \, dV + \frac{1}{2} \int_S \sigma_i u_i \, dS,$$

because the velocity $\partial u_i/\partial t = 0$ outside the interval $[\varepsilon, \tau - \varepsilon]$, as a consequence of the choice we made on t in the definitions of the functions $f_i(\mathbf{x}, t)$, $\sigma_i(\mathbf{x}, t)$ and $u_i(\mathbf{x}, t)$ above. In this way we arrive at the **formula of Clapeyron** [4] from 1834, asserting that *the potential energy of the deformation equals half of the work which the external forces (mass forces and surface forces) would have done, if they had from the beginning the values which they acquire at the deformed equilibrium stage.*

Solving even equilibrium problems of the linear theory of elasticity often brings significant difficulties. This is due primarily to the form of the boundary conditions. The **principle of St. Venant** is helpful in many such situations. *This principle applies to the difference in the stresses and the difference in the deformations inside the body, which result from two different, but statically equivalent systems of surface forces, applied at some portion of the boundary. According to this principle, in domains sufficiently far from this part of the boundary, the difference in the stresses and that in the deformations is ignorably small.*

In 1859 Kirchhoff [19] establishes the uniqueness of the solution to boundary value problems of equilibrium where the stress vector and the displacement are prescribed upon disjoint surfaces S_1 and S_2, respectively, such that the closure of $S_1 + S_2$ is the complete boundary of a finite body V. The displacement $\mathbf{u}$ is determined uniquely to within an infinitesimal rigid displacement. He published these results also in 1876 in [20].

There is a remarkable variational principle enabling us, in the case of equilibrium subject to given surface displacements and vanishing assigned force in the interior, to select among all kinematically possible deformations that one which satisfies the equations of the theory of elasticity, when a positive definite elastic potential is given. The first to recognize its significance was Kelvin, who in 1863 expressed it as "*the elementary condition of stable equilibrium*". As a proved theorem of linear three-dimensional elasticity, it was first given by Love [26] in 1906: *"The displacement that satisfies the equations of equilibrium as well as the conditions at the boundary surface yields a smaller value for the total stored energy that does any other displacement satisfying the same conditions at the bounding surface".*

For a review of the two-dimensional linear elastic problem and that for cylindric bodies the reader is referred to Ivanov [18].

5. THE PROPAGATION OF WAVES

Having given consideration to static problems, let us now consider the propagation of waves.

In Continuum Mechanics waves are described as "singularities" across the two sides of a geometric two-dimensional surface that propagates in space. Such surfaces are called **singular**. To make things specific, consider a family of surfaces given by

$$\mathbf{x} = \mathbf{x}(p_1, p_2, t), \tag{27}$$

where p_1, p_2 are a pair of surface parameters, identifying what we shall call a surface point. The velocity of the surface point, identified in this way, is

$$\left.\frac{\partial \mathbf{x}}{\partial t}\right|_{p_1,p_2=const.}. \tag{28}$$

By eliminating the parameters, we may write (27) in the form

$$\alpha(\mathbf{x}, t) = 0 \tag{29}$$

for some function α. Define the normal component v_n of the velocity of a moving surface by the scalar product

$$v_n \equiv \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{p_1,p_2=const.} \cdot \mathbf{n} = -\frac{\frac{\partial \alpha}{\partial t}}{\sqrt{\alpha_{,i}\,\alpha_{,i}}} \tag{30}$$

where $\mathbf{n}$ is the unit normal to the surface. v_n is called **the speed of displacement** of the surface. The velocity $v_n\mathbf{n}$ is the **normal velocity** of the surface.

Let Ψ be a function defined on the surface, we may for our purposes consider it scalar, vector or tensor-valued. If Ψ undergoes an abrupt change in its value from one side of the surface to the other, the surface is called **a singular surface with respect to the tensor** Ψ. The jump in value of Ψ is denoted by $[\Psi]$. Hugoniot-Duhem theorem states that: *The speed of displacement of a singular surface across which* Ψ *and its derivatives of orders* $1, \dots, p-1$ *are continuous, but at least one p-th derivative of* Ψ *is discontinuous is determined up to sign by the ratio of the jump of* $\partial^p\Psi/\partial t^p$ *to that of the normal p-th derivative,* $\partial^p\Psi/\partial n^p$.

Let us now recall the material representation of a moving surface. If we express the Eulerian coordinates $\mathbf{x}$ via the Lagrangian coordinates $\mathbf{X}$ and substitute them in the definition (29) of the surface, we obtain $S(\mathbf{X}, t) \equiv \alpha(\mathbf{x}(\mathbf{X}, t))$. In the latter representation, which we denote by $S(t)$, we may consider the medium particles as stationary and the surface $S(t)$ moving amongst them, being occupied by a different set of particles at each time t. **The speed of propagation** of the wave is

$$V_N \equiv -\frac{\frac{\partial S}{\partial t}}{\sqrt{S_{,i}\,S_{,i}}}.$$

This speed is a measure of the rate at which the moving surface $S(t)$ traverses the material.

A surface that is singular with respect to some quantity and that has a nonzero speed of propagation is called **a propagating singular surface** or **a wave**.

Above we defined a singular surface with respect to an arbitrary quantity Ψ. Duhem proposed to regard all quantities associated with a motion as functions of the material variables $\mathbf{X}$ and t and to define the **order** of a singular surface with respect to Ψ as the order of the derivative of Ψ of the lowest order suffering a non-zero jump upon the surface.

Some of the most interesting singularities are included in the case when

$$\Psi \equiv \mathbf{x}(\mathbf{X}, t),$$

i.e., are surfaces across which the motion itself, or one of its derivatives, is discontinuous. Surfaces across which at least one of the functional relations $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ or $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$ defining the motion itself is discontinuous are singularities of order zero; those across which some of the first derivatives of $\mathbf{x}$ are discontinuous are of first order, etc.

For a singular surface of order 1, we put $\Psi = x_i$ and obtain

$$[x_{i,k}] = s_i\, N_k \qquad s_i = [N_m\, x_{i,m}], \qquad [\dot{x}_i] = -V_N\, s_i.$$

Here N_k are the components of the unit normal $\mathbf{N}$ to the surface $\mathbf{x} = \mathbf{S}(\mathbf{X}, t)$ defining the motion and equal

$$N_k = \frac{S_{,k}}{\sqrt{S_{,m}\, S_{,m}}}.$$

The vector $\mathbf{N}$ is the **normal velocity** of the material. The vector $\mathbf{s}$, with components s_i, is **the singularity vector**. It is parallel to the jump of velocity, its magnitude varies with the choice of the initial state and thus does not furnish a measure of the strength of the singularity. The jump in the speed of propagation of a singular surface is the negative of the jump in the normal velocity of the material.

For a singular surface of order 2:

$$[x_{i,km}] = s_i\, N_k\, N_m \qquad s_i = [N_k\, N_m\, x_{i,km}]. \tag{31}$$

Also

$$[\dot{x}_{i,k}] = -V_N\, s_i\, N_k, \qquad [\ddot{x}_i] = V_N^2\, s_i. \tag{32}$$

The formulae (31) and (32) show that a singular surface of order 2 is completely determined by a vector $\mathbf{s}$ and the speed of propagation V_N. They show that every wave of second order carries jumps in the velocity gradient and the acceleration. Waves of second order are therefore called **acceleration waves.**

For a body of continuous constant elasticity $\mathbf{C}$, putting $\sigma_{mk} = C_{kmpq}\, \varepsilon_{pq}$ into the equations of motion $\rho\ddot{\mathbf{x}}^k = \sigma_{km,m} + \rho f^k$ yields

$$[\rho\ddot{\mathbf{x}}^k] = C_{kmpq}\, [\mathbf{u}_{p,qm}], \tag{33}$$

where we have supposed that $\rho\mathbf{f}$ is continuous. In linear elasticity $[u_{p,qm}] = \delta_{\alpha q}\delta_{\beta m}[x_{p,\alpha\beta}]$. By applying the general identities (31) and (32) for an acceleration wave, when the present configuration is taken as the initial one, from (33) we obtain

$$\rho\, V^2 s^k = C_{kmpq}\, n_q\, n_m\, s_p, \tag{34}$$

or

$$(C_{kmpq}\, n_q\, n_m\; - \rho\, V^2 \delta_{pk})\, s_p = 0.$$

From this it follows that in order for an acceleration wave with normal $\mathbf{n}$ to exist and propagate, the jump $\mathbf{s}$ which it carries must be an eigenvector of $C_{kmpq}\, n_q\, n_m$ corresponding to the eigenvalue $\rho\, V^2$. For a body such that the work of the stress in any deformation is positive, as in the case for a hyperelastic body with positive definite stored energy, the tensor $C_{kmpq}\, n_q\, n_m$ is positive definite, therefore all eigenvalues $\rho\, V^2$ are positive, and therefore all possible speeds are real. In the general case, *in any linearly elastic body such that the work of the stress is positive for arbitrary deformations, a wave with given normal* $\mathbf{n}$ *may carry a discontinuity of the acceleration parallel to any one of three uniquely determined, mutually orthogonal directions, and corresponding to each of these directions there is a speed of propagation determined uniquely by the elasticities of the material and by* $\mathbf{n}$.

When the eigenvalues $\rho\, V^2$ are not distinct, the above conclusion must be modified, as is seen most easily by considering the isotropic case, for then (33) assumes the more special form

$$[\rho\ddot{\mathbf{x}}^k] = (\lambda + \mu)\,[\mathbf{u}_{p,pk}] + \mu\,[\mathbf{u}_{k,pp}],$$

so that for an acceleration wave we have

$$\rho\, V^2 s_k = (\lambda + \mu)\, s_p n_p n_k + \mu\, s_k,$$

specializing (34). Taking the scalar and vector products of this equation by $\mathbf{n}$ yields

$$(\rho\, V^2 - (\lambda + 2\mu))\,\mathbf{s}\cdot\mathbf{n} = 0, \qquad (\rho\, V^2 - \mu)\,\mathbf{s}\times\mathbf{n} = 0.$$

If $\mathbf{s}\cdot\mathbf{n} \neq 0$, the first of these equations yields $\rho\, V^2 = \lambda + 2\mu$, and the second, if we exclude the case when $\lambda+\mu = 0$, yields $\mathbf{s}\times\mathbf{n} = 0$. If $\mathbf{s}\cdot\mathbf{n} = 0$, but $\mathbf{s}\times\mathbf{n} \neq 0$, the second equation yields $\rho\, V^2 = \mu$. Summarizing these results, we see that *in an isotropic linearly elastic body for which* $\lambda + \mu \neq 0$, *a necessary and sufficient condition that the acceleration waves be propagated at positive speeds is* $\lambda + 2\mu > 0$, $\mu > 0$. *This condition is satisfied when the stored energy is positive definite. Two kinds of acceleration waves are possible: longitudinal waves, whose speed of propagation is given by*

$$V^2 = (\lambda + 2\mu)/\rho,$$

and transverse waves, for which

$$V^2 = \mu/\rho.$$

The foregoing results were first obtained by Christoffel [5] in 1877 and independently by Hugoniot [16] in 1886. These results demonstrate the far-reaching effect of isotropy: instead of three speeds of propagation, for an isotropic body there are only two, but instead of there being only three possible directions for the discontinuity, there are infinitely many, though the possible directions are still far from arbitrary.

6. PERFECT FLUIDS

A continuous medium is a **perfect fluid** if it can support no shearing stress and no couple stress. As a consequence of these restrictions, the stress tensor σ is hydrostatic, $\sigma = -p\mathbf{1}$, and from Cauchy's first law of motion the **dynamical equation of Euler** is obtained

$$\rho\ddot{\mathbf{x}} = -\operatorname{grad} p + \rho\mathbf{f}. \tag{35}$$

Euler published this equation in 1757, see [8]. Cauchy's second law is satisfied automatically, in other words, ballance of linear momentum in a perfect fluid implies balance of moment of momentum, as long as there are no extrinsic couples, while if there are such present, the perfect fluid is incompatible with the principles of mechanics. Hugoniot in 1887 [17] Part I, Hadamard in 1903 [13] and Duhem in 1901 [7] Part II, Chap. IV, proved that a perfect fluid admits only longitudinal waves. Hadamard [13] and Duhem [7] Part II, Chap. I, proved that in an isochoric motion of a perfect fluid wave propagation of any kind is impossible.

In 1869 Kelvin proved that: *"A flow of a perfect fluid subject to lamellar assigned force is circulation preserving if and only if there exists a functional relation*

$$f(p, \rho, t) = 0; \tag{36}$$

alternatively, if and only if, for each fixed time, the pressure is constant, or the density is constant, or the surfaces $p = const.$ coincide with the surfaces $\rho = const.$" Kelvin's theorem is regarded as the fundamental theorem of classical hydrodynamics. Flows satisfying (36) are called **barotropic**.

A perfect fluid may be such that all its flows are barotropic; this is the case for homogeneous incompressible fluids, for which $\rho = const.$ in space and time, and for piezotropic fluids, for which there is an equation of state of the form $p = f(\rho)$. But these conditions are merely sufficient, not necessary for barotropic flow. For example, in a fluid having equations of state $p = F(\rho, \theta) = G(\rho, \eta)$, special conditions may lead to a flow for which $\theta = const.$ or for which $\eta = const.$ Any such flow is barotropic, but the functional form of f in (36) depends upon the particular conditions giving rise to the flow.

When (36) holds, all the numerous theorems appropriate to circulation preserving motion may be applied: the Helmholtz vorticity theorems, the Bernoullian theorems and the Helmholtz theorem of conservation of energy. Indeed, all general theorems of classical hydrodynamics follow from the circulation preserving property.

It should be noted also that *In the case of a barotropic flow, the speed of propagation of acceleration waves*

$$[\ddot{x}_n] = -c^2 \left[\frac{d\log\rho}{dn}\right]$$

is c, where

$$c^2 \equiv \frac{\partial p}{\partial \rho}.$$

This is Hugoniot's theorem, published in 1885, see [15] and [17], Part I.

For barotropic flows in which neither the pressure nor the density is uniform, a necessary and sufficient condition for wave propagation to be possible is that the pressure be an increasing function of the density; this being so, waves of all orders greater than 1 propagate with the unique speed c. Since c is the common speed of propagation of so many kinds of waves, it is called the **speed of sound**.

7. PROBLEMS

In this section we would like to apply the ideas presented so far in order to solve some concrete problems.

Problem 1. A rectangular tank containing a nonviscous liquid of constant density moves horizontally to the right with a constant acceleration. Gravitational force is the only external force. Find the pressure distribution in the liquid and the geometrical shape of the upper surface of the liquid.

Solution. Choose the positive x-direction of the coordinate system to be the direction in which the tank moves, and the positive z-direction to be the vertical direction upward. Then, $d\mathbf{v}/dt = a\,\mathbf{e_1}$ and $\mathbf{b} = -g\,\mathbf{e_3}$, where $a = |d\mathbf{v}/dt|$ is a constant and g is the (constant) acceleration due to gravity. Euler's equation

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{b},$$

where $\mathbf{b}$ is the body force, yields the following three equations for the three Cartesian components (x, y, z) of the ∇p:

$$\frac{\partial p}{\partial x} = -a\rho$$

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} = -g\rho.$$

The second of these tree equations shows that p is independent of y, and thus has the form

$$p = -\rho a x + f(z)\,,$$

where $f(z)$ is an arbitrary function of z. From this form of p and the third component of ∇p above, we see that $f(z) = -\rho g z + C$, where C is a constant, thus arriving at

$$p = -\rho(ax + gz) + C.$$

At the point where the z-axis meets the upper surface of the liquid, we have $p = p_a$, where p_a is the atmospheric pressure. If this point is at a height h above the origin, the last equation for p gives $C = p_a + \rho g h$. Thus the pressure distribution of the liquid is

$$p = p_a - \rho(ax + gz - gh).$$

For $p = p_a$, the last equation for p becomes

$$z = -\left(\frac{a}{g}\right)x + h.$$

This is the shape of the upper surface of the liquid. Evidently, this surface is a plane, making an acute angle $\theta = \tan^{-1}(a/g)$ with the horizontal. In the limiting case when $a \to 0$, the liquid moves with a constant velocity and the upper surface of the liquid becomes a horizontal plane.

The interested reader is invited to apply the method of solution of the last problem in order to solve

Problem 2. A column of a nonviscous liquid of constant density contained in a vertical circular vessel rotates like a rigid body about the axis of the vessel with a constant angular velocity ω. Gravitational force is the only external force. Find the pressure distribution in the liquid and the geometrical form of the upper surface of the liquid.

In the next problem we will use Bernoulli's equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} = -\nabla H\,,$$

where $\mathbf{w}$ is the vorticity vector and $H \equiv P + \chi + v^2/2$. Here P is

$$P = \int \frac{1}{\rho}\, dp\,,$$

with p being the pressure. This equation is known after Daniel Bernoulli (1738). It is the equation of motion for an elastic fluid moving under conservative body force $-\nabla\chi$. Since Bernoulli's equation holds for an elastic fluid for which $\rho = \rho(p)$, it automatically holds in the special case of $\rho = \mathit{constant}$.

Problem 3. For a certain flow of a nonviscous fluid of constant density under the Earth's gravitational field, the velocity distribution is given by $\mathbf{v} = \nabla\phi$, where $\phi = x^3 - 3xy^2$. Find the pressure distribution.

Solution. From the given $\mathbf{v}$, we find that curl $\mathbf{v} = \mathbf{0}$ and $\partial\mathbf{v}/\partial t = \mathbf{0}$. Thus the fluid is irrotational and steady. Then $\partial\mathbf{v}/\partial t = \mathbf{0}$ and either the vorticity vector $\mathbf{w} = \mathbf{0}$ or $\mathbf{v} \times \mathbf{w} = \mathbf{0}$. Further, since the body force is the gravitational force, it is conservative. With these observations, Bernoulli's equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} = -\nabla H,$$

where H is the Bernoulli's function $H \equiv P + \chi + v^2/2$, reduces to

$$\nabla H = \mathbf{0}.$$

Since $\partial H/\partial t = 0$, it follows that

$$H \equiv P + \chi + \frac{v^2}{2} = constant$$

everywhere in the fluid. Thus, under the assumed conditions, the function H is an integral of the equation of motion of the elastic fluid.

Since the body force is conservative, $\chi = gz$, where z is measured vertically upward. Accordingly, from $H = P + \chi + v^2/2 = constant$ with $P = p/\rho$ we obtain

$$\frac{p}{\rho} + \frac{1}{2}v^2 + gz = C,$$

where C is a constant.

From the given $\mathbf{v}$ we also find

$$v_1 = \frac{\partial \phi}{\partial x} = 3(x^2 - y^2), \qquad v_2 = \frac{\partial \phi}{\partial y} = -6xy, \qquad v_3 = 0$$

and hence

$$v^2 = v_1^2 + v_2^2 = 9(x^2 + y^2)^2.$$

Substituting this result into the equation relating v^2, p and z (above), we obtain

$$\frac{p}{\rho} + \frac{9}{2}(x^2 + y^2)^2 + gz = C.$$

From this result it is evident that $C = p^0/\rho$, where p^0 is the pressure at the origin. Thus,

$$p = p^0 - \rho\Big(\frac{9}{2}(x^2 + y^2)^2 + gz\Big)$$

is the sought pressure distribution.

Many interesting problems in Continuum Mechanics can be fond in the book of Chandrasekharaiah and Debnath [3].

8. LINEARLY VISCOUS FLUIDS

Let us now consider a medium which in equilibrium, satisfies Euler's equation (35)

$$\operatorname{grad} p = \rho\, \mathbf{f},$$

but when in motion can support appropriate shearing stresses. More specifically, let us assume that the stress tensor σ is a linear function of the velocity $\dot{\mathbf{x}}$ and the velocity gradient, namely

$$\sigma = \mathbf{g}(\dot{\mathbf{x}}, \mathbf{w}, \mathbf{d}), \tag{37}$$

where $\mathbf{g}$ is a linear function. Here $\mathbf{d}$ is Euler's stretching tensor and $\mathbf{w}$ is Cauchy's spin tensor. The constitutive equations (37) define **linearly viscous fluid**. By applying the principle that the constitutive equation must have the same form for all observers, one shows that in fact σ is independent of $\dot{\mathbf{x}}$ and $\mathbf{w}$, i.e.,

$$\sigma = \mathbf{g}(\mathbf{d}), \qquad (38),$$

with the function $\mathbf{g}$ being linear. This equation in an internal frame, along with $\mathbf{f}(0) = -p\,\mathbf{1}$, was taken as the definition of a fluid by Stokes [35] in 1845. If we now use a coordinate system with axes that coincide with the principal directions of $\mathbf{d}$, so that (38) becomes

$$\sigma_{km} = f_{km}(d_1, d_2, d_3)$$

it is easily seen that *the principal axes of stretching are also principal axes of stress.* Another interesting property of fluids included in the definition (37) is that such fluids are necessarily isotropic.

The most general linear isotropic function σ of a symmetric second order tensor $\mathbf{d}$ may be written in the form of **Navier-Poisson law**:

$$\sigma = -p\,\mathbf{1} + \lambda\,\mathbf{I_d}\,\mathbf{1} + 2\mu\mathbf{d}$$

or in components

$$\sigma_{km} = -p\,\delta_{km} + \lambda\,d_{qq}\delta_{km} + 2\mu\,d_{km}\,, \tag{39}$$

where a use is made of the requirement that $\sigma = -p\,\mathbf{1}$ when $\mathbf{d} = 0$. Historically, the simplest case of this law was proposed by Newton [31], Lib. II, Chap.IX. It follows from (39) that σ is symmetric, thus Cauchy's second law is automatically satisfied. Thus, for the fluids in question, ballance of momentum implies ballance of moment of momentum. Substitution of (39) into Cauchy's first law of motion

$$\sigma_{ij,j} + \rho f_i = \rho\frac{dv_i}{dt}, \qquad i = 1, 2, 3$$

yields a system of three differential equations, known, when subjected to further simplifying assumptions, as **the Navier-Stokes equations**:

$$\mu\nabla^2\mathbf{v} + (\lambda + \mu)\nabla(\operatorname{div}\mathbf{v}) - \nabla p + \rho\mathbf{f} = \rho\frac{d\mathbf{v}}{dt}. \tag{40}$$

They are attributed to Navier (1822) and Stokes (1845) and hold for both compressible and incompressible viscous fluid flows; in the incompressible case $\rho = \rho_0$ and $\operatorname{div}\mathbf{v} = 0$.

The coefficients λ and μ are the **viscosities** of the fluid. In the absence of viscosity, that is if λ and μ are negligibly small, this equation reduces to Euler's equation of motion (35) for perfect fluids. Because of that perfect fluids are often called **inviscid**.

The portion $\lambda\, \mathbf{I_d}\, \mathbf{1} + 2\mu\mathbf{d}$ of the stress is considered as arising from internal friction. Because

$$\sigma_{km} = 2\mu\, d_{km} \quad \text{when} \quad k \neq m, \tag{41}$$

μ is the ratio of the shear stress to the corresponding shearing of any two orthogonal elements, and so is called the **shear viscosity**.

The stress power assumes the form

$$P = \sigma_{km}\, d_{km} = -p\, d_{kk} + \lambda(d_{kk})^2 + 2\mu\, d_{km}\, d_{mk}.$$

In 1850 Stokes had shown in [36] that for the fluids in consideration

$$\mu \geq 0, \qquad 3\lambda + 2\mu \geq 0.$$

The same conclusion was reached independently by Duhem in 1901, published in [7], Part I. These inequalities have some significant mechanical consequences. For example, equations (40) with $\mu \geq 0$ imply that the shear stress always opposes the shearing. These consequences show that the effect of the viscous stress $\sigma + p\mathbf{1}$, as given by (39) is always to resist change of shape, and thus is of the nature of frictional resistance.

For an incompressible viscous fluid, the Navier-Stokes equations (40) are rewritten in the form:

$$\frac{\mu}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\nabla p + \mathbf{f} = \frac{d\mathbf{v}}{dt}. \tag{42}$$

The coefficient μ/ρ is called kinematic viscosity.

The presence of viscosity has the effect of making the propagation of most kinds of waves impossible. In 1926 Kotchine [21] proved that the instantaneous existence of a surface upon which $\dot{\mathbf{x}}$ and p are continuous but $\dot{x}_{k,m}$ suffers a jump discontinuity is incompatible with the law of linear viscosity (39). His result is contained in an earlier one of Duhem from 1901 [7], Part II, Chap. III, who uses a different terminology. In [6] and [7], Part II, Chap. III, Duhem asserts that in a linearly viscous fluid no waves of order greater than 1 are possible.

A summary of the existing knowledge of the theories of non-linear viscosity is given in [38]. An excellent text from the latter part of the 20th Century, which the reader can use to get acquainted with the modern developments of the presented theories, is the book of Timoshenko and Goodier [37].

For the readers privileged to know Russian, we list two excellent texts on hydrodynamics [40], [41]. They can be used to deepen knowledge in the theories presented in this review. Two prominent texts in Bulgarian on hydrodynamics are the book of Zaprianov and that of Shkadov and Zaprianov, [39] and [42].

9. CONDITIONS FOR COMPATABILITY OF THE STRESSES

The conditions for compatability of the stresses are due to Beltrami [1], and were independently discovered by Mitchell in 1899 [27].

We consider the static case in which the equations of motion

$$\sigma_{ij,j} + \rho f_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

become the **equations for equilibrium**

$$\sigma_{ij,j} + F_i = 0, \tag{43}$$

where $\mathbf{F} = r\mathbf{f}$. These equations form a system of 3 equations for the 6 unknowns σ_{ij} (we assume that the volume forces $\mathbf{f}$ are given). This system has infinitely many solutions, but not every one of them corresponds to a real deformation, from which we can calculate the displacement $\mathbf{u}$ in the medium. As we know, for the deformations, determined using equation (17), it is necessary and sufficient to satisfy the conditions for the compatability of the deformations of St. Venant

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0. \tag{44}$$

Let us express these conditions with the stresses. Let's substitute the components of the tensor of deformations, using equations (17), into the conditions (44), and then introduce the notation $\sigma = \sigma_{kk}$. We obtain

$$\sigma_{ij,kl} + \sigma_{kl,ij} - \sigma_{ik,jl} - \sigma_{jl,ik} = \frac{\nu}{1+\nu}(\sigma_{,ij}\delta_{kl} + \sigma_{,kl}\delta_{ij} - \sigma_{,ik}\delta_{jl} - \sigma_{,jl}\delta_{ik}). \tag{45}$$

If we set $k = l$ and sum over the repeated index, we will arrive at the following system of equations:

$$\sigma_{ij,kk} + \sigma_{,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} = \frac{\nu}{1+\nu}(\sigma_{,ij} + \sigma_{,kk}\delta_{ij}). \tag{46}$$

This system consists of 9 equations, from which independent are only 6. We can obtain these 6 independent equations if we let, for example, $i \geq j$, because of the symmetry with respect to the indexes i and j. The system (46), obtained in this manner, is equivalent to the initial system (45), because each system consists of 6 independent equations, and the equations of system (46) are linear combinations of the equations of system (45).

Let us differentiate the equations (43) with respect to x_k. We obtain

$$\sigma_{ij,jk} = -F_{i,k}. \tag{47}$$

Substitute (47) in (46) to obtain

$$\sigma_{ij,kk} + \frac{\nu}{1+\nu}\sigma_{,ij} - \frac{\nu}{1+\nu}\sigma_{,kk}\,\delta_{ij} = -(F_{i,j} + F_{j,i}). \tag{48}$$

We are now going to simplify this system in the following way: We set in (45) $k = i$ and $l = j$ and after a few calculations obtain

$$\sigma_{ij,ij} = \frac{1-\nu}{1+\nu}\sigma_{,ii}. \tag{49}$$

Now using (47), we can write equation (49) as

$$\sigma_{,ii} = -\frac{1+\nu}{1-\nu}F_{i,i}.$$

Substituting this result in (48), we finally obtain **the conditions of Beltrami-Mitchell for the compatability of the stresses**:

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{,ij} = -\frac{\nu}{1-\nu}F_{k,k}\,\delta_{ij} - (F_{i,j} + F_{j,i}).$$

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