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REPRESENTATION OF NATURAL NUMBERS BY SUMS OF FOUR SQUARES OF ALMOST-PRIME HAVING A SPECIAL FORM

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In this paper we consider the equation $x_1^2+x_2^2+x_3^2+x_4^2=N$, where N is a sufficiently large integer and prove that if η is quadratic irrational number and $0<\lambda<\frac{1}{10}$, then it has a solution in almost-prime numbers $x_1,\dots,x_4$, such that $\{\eta x_i\}<N^{-\lambda}$ for $i=1,\dots,4$.

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1. INTRODUCTION AND STATEMENT OF THE RESULT

In 1770 Lagrange proved that for any positive integer N the equation

$$x_1^2+x_2^2+x_3^2+x_4^2=N \tag{1.1}$$

has a solution in integer numbers $x_1,\dots,x_4$. Later Jacobi found an exact formula for the number of the solutions (see [8, Ch. 20]). A lot of researchers studied the equation (1.1) for solvability in integers satisfying additional conditions. There is a hypothesis stating that if N is sufficiently large and $N\equiv 4 \pmod{24}$ then (1.1) has a solution in primes. This hypothesis has not been proved so far, but several approximations to it have been established.

In 1994 J. Brüdern and E. Fouvry [1] proved that for any large $N \equiv 4 \pmod{24}$, the equation (1.1) has a solution in $x_1, \ldots, x_4 \in \mathcal{P}_{34}$. (We say that integer n is an almost-prime of order r if n has at most r prime factors, counted with their multiplicities. We denote by $\mathcal{P}_r$ the set of all almost-primes of order r.) This result was improved by D. R. Heath-Brown and D. I. Tolev [9]. They showed that, under the same restrictions for N, the equation (1.1) has a solution in prime x_1 and almost-prime $x_2, x_3, x_4 \in \mathcal{P}_{101}$. In their paper they also proved that the equation has a solution in $x_1, \ldots, x_4 \in \mathcal{P}_{25}$. In 2020 Tak Wing Ching [2] improved this result with three of them being in $\mathcal{P}_3$ and the other in $\mathcal{P}_4$.

On the other hand, let us consider a subset of the set of integers having the form

$$\mathcal{A} = \{n \,|\, a < \{\eta n\} < b\},$$

where η is a fixed quadratic irrational number, and $a, b \in [0, 1]$.

Denote by $I(N)$ the number of solutions of (1.1) in arbitrary integers and by $J(N)$ the number of solutions of (1.1) in integers from the set $\mathcal{A}$.

In 2011 S. A. Gritsenko and N. N. Motkina [6] proved that for any positive small ε, the following formula holds

$$J(N) = (b - a)^4 I(N) + O\left(N^{0,9+3\varepsilon}\right).$$

S. A. Gritsenko and N. N. Motkina consider many others additive problem in witch variables are in special set of numbers similar to $\mathcal{A}$. (See [4] – [5] and [7].) In 2013 A. V. Shutov [12] considered solvability of diophantine equation in integer numbers from $\mathcal{A}$. Further research in this area was made by A. V. Shutov and A. A. Zhukova [13].

We consider the equation (1.1), where x_i are almost-prime numbers and belong to a set similar to $\mathcal{A}$. Our result is

Theorem 1.1. *Let η be a quadratic irrational number, $0 < \lambda < \frac{1}{10}$ and $k = \left[\frac{54}{1-10\lambda}\right]$. Then for every sufficiently large integer N, the equation* (1.1) *has a solution in almost-prime numbers $x_1, \ldots, x_4 \in \mathcal{P}_k$, such that $\{\eta x_i\} < N^{-\lambda}$, $i = 1, 2, 3, 4$.*

In the present paper we use the following notations.

We denote by N a sufficiently large odd integer and $P = N^{\frac{1}{2}}$. Letters a, b, k, l, m, n, q, p always stand for integers. By $(n_1, \ldots, n_k)$ we denote the greatest common divisor of $n_1, \ldots, n_k$. Let $||t||$ denote the distance from t to the nearest integer. We denote by $\vec{n}$ four dimensional vectors and let

$$|\vec{n}| = \max(|n_1|, \ldots, |n_4|). \tag{1.2}$$

As usual, $\mu(q)$ is the Möbius function and $\tau(q)$ is the number of positive divisors of q. Sometimes we write $a \equiv b\ (q)$ as an abbreviation of $a \equiv b \pmod{q}$.

We write $\sum\limits_{x\ (q)}$ for a sum over a complete system of residues modulo q and respectively $\sum\limits_{x\ (q)}^{\ *}$ is a sum over a reduced system of residues modulo q. We also denote $e(t) = e^{2\pi i t}$.

We use Vinogradov's notation $A \ll B$, which is equivalent to $A = O(B)$. By ε we denote an arbitrarily small positive number, which is not the same in different occurrences. The constants in the O-terms and $\ll$-symbols are absolute or depend on ε.

2. AUXILIARY RESULTS

Now we introduce some lemmas, which shall be used later.

Lemma 2.1. *Suppose that $D \in \mathbb{R}, D > 4$. There exist arithmetical functions $\lambda^{\pm}(d)$ (called Rosser's functions of level D) with the following properties:*

1. *For any positive integer d we have*

$$|\lambda^{\pm}(d)| \leq 1, \qquad \lambda^{\pm}(d) = 0 \quad \text{if} \quad d > D \quad \text{or} \quad \mu(d) = 0.$$

2. *If $n \in \mathbb{N}$ then*

$$\sum_{d|n} \lambda^{-}(d) \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda^{+}(d).$$

3. *If $z \in \mathbb{R}$ is such that $z^2 \leq D$ and if*

$$P(z) = \prod_{2<p<z} p, \ \ \mathcal{B} = \prod_{2<p<z} \left(1 - \frac{1}{p-1}\right), \ \ \mathcal{N}^{\pm} = \sum_{d|P(z)} \frac{\lambda^{\pm}(d)}{\varphi(d)}, \ \ s_0 = \frac{\log D}{\log z}, \tag{2.1}$$

then we have

$$\mathcal{B} \leq \mathcal{N}^{+} \leq \mathcal{B}\left(F(s_0) + O\left((\log D)^{-\frac{1}{3}}\right)\right), \tag{2.2}$$

$$\mathcal{B} \geq \mathcal{N}^{-} \geq \mathcal{B}\left(f(s_0) + O\left((\log D)^{-\frac{1}{3}}\right)\right), \tag{2.3}$$

where $F(s)$ and $f(s)$ satisfy

$$\begin{aligned} &F(s) = 2e^{\gamma} s^{-1}, \quad \text{if} \quad 2 \leq s \leq 3,\\ &f(s) = 2e^{\gamma} s^{-1} \log(s-1), \quad \text{if} \quad 2 \leq s \leq 3,\\ &(sF(s))' = f(s-1), \quad \text{if } s > 3,\\ &(sf(s))' = F(s-1), \quad \text{if } s > 2. \end{aligned}$$

Here γ is Euler's constant.

Proof. See Greaves [3, Chapter 4]. □

Lemma 2.2. *Suppose that* $\Lambda_i, \Lambda_i^{\pm}$ *are real numbers satisfying* $\Lambda_i = 0$ *or* 1, $\Lambda_i^- \le \Lambda_i \le \Lambda_i^+$, $i = 1, 2, 3, 4$. *Then*

$$\begin{aligned}\Lambda_1\Lambda_2\Lambda_3\Lambda_4 \ge &\Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+\Lambda_4^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^-\Lambda_4^+ + \\ &+ \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^- - 3\Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+.\end{aligned} \tag{2.4}$$

Proof. The proof is similar to the proof of [1, Lemma 13]. □

Let

$$w_0(t) = \begin{cases} e^{\frac{1}{t^2 - \frac{16}{25}}} & \text{if } t \in \left(-\frac{4}{5}, \frac{4}{5}\right), \\ 0 & \text{if } t \notin \left(-\frac{4}{5}, \frac{4}{5}\right) \end{cases}$$

and

$$w(x) = w_0\left(\frac{x}{P} - \frac{1}{2}\right). \tag{2.5}$$

Lemma 2.3. *Let* $u, \beta \in \mathbb{R}$ *and*

$$J(\beta, u) = \int_{-\infty}^{+\infty} w_0\left(x - \frac{1}{2}\right) e(\beta x^2 + ux)dx. \tag{2.6}$$

Then:

1. *For every* $k \in \mathbb{N}$ *and* $u \neq 0$ *we have*

$$J(\beta, u) \ll_k \frac{1 + |\beta|^k}{|u|^k}.$$

2. *The following inequality hold*

$$J(\beta, u) \ll \min\left(1, |\beta|^{-\frac{1}{2}}\right).$$

Proof. See [9, Lemma 9]. □

Lemma 2.4. *Suppose that* $\vec{u} \in \mathbb{Z}^4$ *and*

$$J(\beta, \vec{u}) = \prod_{i=1}^{4} J(\beta, u_i).$$

Then we have

$$\int_{-\infty}^{+\infty} |J(\beta, \vec{u})|\, d\gamma \ll |\vec{u}|^{-1+\varepsilon}.$$

Proof. Proof can be find in [9, Lemma 10]. □

Lemma 2.5. *There exists a function* $\sigma(\upsilon, q, \gamma)$ *defined for* $-\frac{q}{2} < \upsilon \leq \frac{q}{2}$, $q \leq P$, $|\gamma| \leq \frac{P}{q}$, *integrable with respect to* γ, *satisfying*

$$|\sigma(\upsilon, q, \gamma)| \leq \frac{1}{1+|\upsilon|}$$

and also for every $a \in \mathbb{Z}$, $(a, q) = 1$ *we have*

$$\sum_{-\frac{q}{2}<\upsilon\leq\frac{q}{2}} e\left(\frac{\overline{a}\upsilon}{q}\right)\sigma(\upsilon, q, \gamma) = \begin{cases} 1 & \text{if } \gamma \in \mathcal{N}(a,q), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mathcal{N}(a, q) = \left(-\frac{P^2}{q(q+q')}, \frac{P^2}{q(q+q'')}\right]$$

and

$$P < q + q', q + q'' \leq P + q, \qquad aq' \equiv 1(\text{mod} q), \qquad aq'' \equiv -1(\text{mod} q). \tag{2.7}$$

Proof. See [15, Lemma 45]. □

For $q \in \mathbb{N}$ and $m, n \in \mathbb{Z}$, the Gauss sum is defined by

$$G(q, m, n) = \sum_{x(q)} e\left(\frac{mx^2 + nx}{q}\right). \tag{2.8}$$

For $\vec{d} = \langle d_1, \dots, d_4\rangle \in \mathbb{Z}^4$ and $\vec{n} = \langle n_1, \dots, n_4\rangle \in \mathbb{Z}^4$ we denote

$$G(q, a\vec{d}^2, \vec{n}) = \prod_{i=1}^{4} G(q, ad_i^2, n_i).$$

We need to estimate an exponential sum of the form

$$V_q = V_q(N, \vec{d}, \upsilon, \vec{n}) = \sum_{a(q)}{}^{*} e\left(\frac{\overline{a}\upsilon - Na}{q}\right) G(q, a\vec{d}^2, \vec{n}). \tag{2.9}$$

To estimate V_q we use the properties of the Gauss sum and the Kloosterman sum.

Lemma 2.6. *Suppose that* $N, q \in \mathbb{N}, \upsilon \in \mathbb{Z}$ *and* $\vec{d}, \vec{n} \in \mathbb{Z}^4$. *Then we have*

$$V_q(N, \vec{d}, \upsilon, \vec{n}) \ll q^{\frac{5}{2}}\tau(q)(q, N)^{\frac{1}{2}}(q, d_1)(q, d_2)(q, d_3)(q, d_4).$$

Moreover, if some of the conditions

$$(q, d_i)|n_i, \quad i = 1, \dots, 4$$

do not hold, then $V_q(N, \vec{d}, \upsilon, \vec{n}) = 0$.

Proof. This result is analogous to this one in [1, Lemma 1]. □

Lemma 2.7. *(Liouville) If η is an irrational number which is the root of a polynomial f of degree 2 with integer coefficients, then there exists a real number $A > 0$ such that, for all integers p, q, with $q > 0$,*

$$\left|\eta - \frac{p}{q}\right| \geq \frac{A}{q^2}.$$

Proof. See [11, Theorem 1A]. □

3. PROOF OF THE THEOREM

3.1. BEGINNING OF THE PROOF

Let N be a sufficiently large integer. We denote

$$z = N^{\alpha}, \qquad P(z) = \prod_{p<z} p, \qquad \delta = N^{-\lambda}.$$

We apply the well-known Vinogradov's "little cups" lemma (see [10, Chapter 1, Lemma A]) with parameters

$$\alpha_1 = \frac{\delta}{4}, \qquad \beta_1 = \frac{3\delta}{4}, \qquad \Delta = \frac{\delta}{2}, \qquad r = [\log N]$$

and construct a function $\theta(t)$ which is periodic with period 1 and has the following properties:

$$\theta\left(\frac{\delta}{2}\right) = 1; \quad 0 < \theta(t) < 1 \quad \text{for} \quad 0 < t < \frac{\delta}{2} \quad \text{or} \quad \frac{\delta}{2} < t < \delta;$$

$$\theta(t) = 0 \quad \text{for} \quad \delta \leq t \leq 1.$$

Furthermore, from the Fourier series of $\theta(t)$ we find

$$\theta(t) = \frac{\delta}{2} + \sum_{\substack{0<|m|\leq H \\ m\neq 0}} c(m)\, e(mt) + O(P^{-A}), \tag{3.1}$$

with

$$|c(m)| \leq \min\left(\frac{\delta}{2}, \frac{1}{|m|}\left(\frac{[\log N]}{\delta\pi|m|}\right)^{[\log N]}\right),$$

where A is arbitrary large constant and

$$H = \frac{[\log N]^2}{\delta}. \tag{3.2}$$

Let us denote

$$\theta(\eta\vec{x}) = \theta(\eta x_1)\theta(\eta x_2)\theta(\eta x_3)\theta(\eta x_4)$$

and

$$w(\vec{x}) = w(x_1)w(x_2)w(x_3)w(x_4).$$

We consider the sum

$$\Gamma = \sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=N \\ (x_i,P(z))=1,\ i=1,2,3,4}} \theta(\eta\vec{x})w(\vec{x}).$$

From the condition $(x_i, P(z)) = 1$ it follows that any prime factor of x_i is greater than or equal to z. Suppose that x_i has l prime factors, counted with their multiplicities. Then we have

$$N^{\frac{1}{2}} \geq x_i \geq z^l = N^{\alpha l}$$

and hence $l \leq \frac{1}{2\alpha}$. This implies that if $\Gamma > 0$ then equation (1.1) has a solution in almost-prime numbers $x_1, \dots, x_4$ with at most $\left[\frac{1}{2\alpha}\right]$ prime factors, such that $\{\eta x_i\} < N^{-\lambda}$, $i = 1, \dots, 4$.

For $i = 1, 2, 3, 4$ we define

$$\Lambda_i = \sum_{d|(x_i,P(z))} \mu(d) = \begin{cases} 1 & \text{if } (x_i, P(z)) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Then we find that

$$\Gamma = \sum_{x_1^2+x_2^2+x_3^2+x_4^2=N} \Lambda_1\Lambda_2\Lambda_3\Lambda_4\theta(\eta\vec{x})w(\vec{x}).$$

We can write Γ as

$$\Gamma = \sum_{x_i\in\mathbb{Z}} \Lambda_1\Lambda_2\Lambda_3\Lambda_4\theta(\eta\vec{x})w(\vec{x}) \int_0^1 e(\alpha(x_1^2 + x_2^2 + x_3^2 + x_4^2 - N))\, d\alpha.$$

Suppose that $\lambda^{\pm}(d)$ are the Rosser functions of level D (see Lemma 2.1). Let also denote

$$\Lambda_i^{\pm} = \sum_{d|(x_i,P(z))} \lambda^{\pm}(d), \qquad i = 1, 2, 3, 4. \tag{3.4}$$

Then from Lemma 2.1, (3.3) and (3.4) we find that

$$\Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+.$$

We use Lemma 2.2 and find that

$$\Gamma \geq \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 - 3\Gamma_5,$$

where $\Gamma_1, \ldots, \Gamma_5$ are the contributions coming from the consecutive terms of the right side of (2.4). We have $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4$ and

$$\Gamma_1 = \sum_{x_i \in \mathbb{Z}} \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \theta(\eta \vec{x}) w(\vec{x}) \int_0^1 e(\alpha(x_1^2 + x_2^2 + x_3^2 + x_4^2 - N)) d\alpha,$$

$$\Gamma_5 = \sum_{x_i \in \mathbb{Z}} \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \theta(\eta \vec{x}) w(\vec{x}) \int_0^1 e(\alpha(x_1^2 + x_2^2 + x_3^2 + x_4^2 - N)) d\alpha.$$

Hence, we get

$$\Gamma \geq 4\Gamma_1 - 3\Gamma_5. \tag{3.5}$$

3.2. ASYMPTOTIC FORMULA FOR Γ_1

We shall find an asymptotic formula for the integral Γ_1. We have

$$\begin{aligned}\Gamma_1 = \sum_{d_i \mid P(z)} \lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)\lambda^+(d_4) \sum_{x_i \equiv 0(d_i)} \theta(\eta \vec{x}) w(\vec{x}) \times \\ \times \int_0^1 e(\alpha(x_1^2 + \cdots + x_4^2 - N)) d\alpha \\ = \sum_{d_i \mid P(z)} \lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)\lambda^+(d_4) \times \\ \times \int_0^1 \prod_{1 \leq i \leq 4} \Bigg(\sum_{x \equiv 0(d_i)} \theta(\eta x) w(x) e(\alpha x^2) \Bigg) e(-N\alpha) d\alpha.\end{aligned}$$

Let

$$S(\alpha, d, m) = \sum_{\substack{x \in \mathbb{Z} \\ x \equiv 0(d)}} w(x) e(\alpha x^2 + m\eta x)\,. \tag{3.6}$$

Then using the Fourier series of $\theta(t)$ (see (3.1)), we find

$$\sum_{x \equiv 0(d)} \theta(\eta x) w(x) e(\alpha(x^2) = \sum_{|m| \leq H} c(m) \sum_{x \equiv 0(d)} w(x) e(\alpha x^2 + m\eta x) + O\big(P^{-A}\big)\,.$$

Denoting

$$S(\alpha, \vec{d}, \vec{m}) = S(\alpha, d_1, m_1) S(\alpha, d_2, m_2) S(\alpha, d_3, m_3) S(\alpha, d_4, m_4) \tag{3.7}$$

and

$$\lambda(\vec{d}) = \lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)\lambda^+(d_4)\,, \tag{3.8}$$

we find that

$$\Gamma_1 = \sum_{d_i|P(z)} \lambda(\vec{d}) \sum_{\substack{|m_i|\le H \\ i=1,2,3,4}} c(m_i) \int_0^1 S(\alpha,\vec{d},\vec{m})e(-N\alpha)d\alpha + O(1).$$

We divide Γ_1 into two parts:

$$\Gamma_1 = \Gamma_1^0 + \Gamma_1^* + O(1),$$

where

$$\Gamma_1^0 = c^4(0) \sum_{d_i|P(z)} \lambda(\vec{d}) \sum_{\substack{x_i\equiv 0(d_i) \\ x_1^2+x_2^2+x_3^2+x_4^2=N}} w(\vec{x})$$

and

$$\Gamma_1^* = \sum_{d_i|P(z)} \lambda(\vec{d}) \sum_{\substack{0<|m_i|\le H \\ i=1,2,3,4}} c(m_i) \int_0^1 S(\alpha,\vec{d},\vec{m})e(-N\alpha)\,d\alpha\,. \tag{3.9}$$

Hence

$$\Gamma \ge 4\Gamma_1^0 - 3\Gamma_5^0 + O\big(\Gamma_1^*\big) + O\big(\Gamma_5^*\big) + O(1). \tag{3.10}$$

According to [1] and [9], for $D \le P^{1/8-\varepsilon}$, $s = \dfrac{\log D}{\log z} = 3.13$ the estimate

$$4\Gamma_1^0 - 3\Gamma_5^0 \gg \frac{C\delta N}{(\log N)^4} + O\big(\delta P^{3/2+\varepsilon} D^4\big) \tag{3.11}$$

with some constant C is obtained. Thus it suffices to evaluate Γ_1^* and Γ_5^*.

3.3. ESTIMATION OF Γ_1^*

In this subsection we find the upper bound for Γ_1^* defined in (3.9). The function in the integral in Γ_1^* is periodic with period 1, so we can integrate over the interval $\mathcal{I}$ defined as

$$\mathcal{I} = \left(\frac{1}{1+[P]}, 1+\frac{1}{1+[P]}\right).$$

We apply the Kloosterman form of the Hardy-Littlewood circle method. We divide the interval only into large arcs. Using the properties of the Farey fractions, we represent $\mathcal{I}$ as an union of disjoint intervals in the following way:

$$\mathcal{I} = \bigcup_{q\le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathcal{L}(a,q),$$

where

$$\mathcal{L}(a,q) = \left(\frac{a}{q} - \frac{1}{q(q+q')}, \frac{a}{q} + \frac{1}{q(q+q'')}\right]$$

and where the integers q', q'' are specified in (2.7). Then

$$\Gamma_1^* = \sum_{d_i | P(z)} \lambda(\vec{d}) \sum_{\substack{0<|m_i|\le H\\ i=1,2,3,4}} c(m_i) \sum_{q\le P} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathcal{L}(a,q)} S(\alpha, \vec{d}, \vec{m}) e(-N\alpha)\, d\alpha.$$

We change variable of integration $\alpha = \frac{a}{q} + \beta$ to get

$$\Gamma_1^* = \sum_{d_i | P(z)} \lambda(\vec{d}) \sum_{\substack{0<|m_i|\le H\\ i=1,2,3,4}} c(m_i) \sum_{q\le P} \sum_{\substack{a=1\\(a,q)=1}}^{q} \times$$
$$\times \int_{\mathcal{M}(a,q)} S\left(\frac{a}{q} + \beta, \vec{d}, \vec{m}\right) e\left(-N\left(\frac{a}{q}+\beta\right)\right) d\beta,$$

where

$$\mathcal{M}(a,q) = \left(-\frac{1}{q(q+q')}, \frac{1}{q(q+q'')}\right].$$

From (2.7) we find that

$$\left[-\frac{1}{2qP}, \frac{1}{2qP}\right] \subset \mathcal{M}(a,q) \subset \left[-\frac{1}{qP}, \frac{1}{qP}\right]$$

and hence

$$|\beta| \le \frac{1}{qP} \quad \text{for} \quad \beta \in \mathcal{M}(a,q). \tag{3.12}$$

Now we consider the sum $S(\alpha, d_i, m_i)$ defined in (3.6). As η is irrational number, $||s\eta|| \ne 0$ for all $s \in \mathbb{Z}$. Using that fact and working as in the proof of [9, Lemma 12], we find that for $\beta \in \mathcal{M}(a,q)$ we have

$$S\left(\frac{a}{q} + \beta, d_i, m_i\right) = \frac{P}{d_i q} \sum_{|n - m_i d_i q\eta| < M_i} J\left(\beta P^2,\ (m_i\eta - \frac{n}{d_i q})P\right) G(q,\ a d_i^2,\ n) +$$
$$+ O(P^{-B}), \tag{3.13}$$

where $G(q,m,n)$ and $J(\gamma,u)$ are defined respectively by (2.8) and (2.6), B is an arbitrarily large constant, $M_i = d_i P^{\varepsilon}$, $\varepsilon > 0$ is arbitrarily small and the constant in the O-term depends only on B and ε. We leave the verification of the last formula to the reader.

Let

$$F(P, \vec{d}) = \sum_{\substack{0<|m_i|\le H\\ i=1,2,3,4}} c(m_i) \sum_{q\le P} \sideset{}{^*}\sum_{a\,(q)} e\left(-\frac{aN}{q}\right) \int_{\mathcal{M}(a,q)} S\left(\frac{a}{q} + \beta, \vec{d}, \vec{m}\right) e(-\beta N) d\beta.$$

It is obvious that

$$\Gamma_1^* = \sum_{d_i | P(z)} \lambda(\vec{d}) F(P, \vec{d})\,. \tag{3.14}$$

Using (3.13) and Lemma 2.3 we get

$$F(P,\, \vec{d}) = F^*(P,\, \vec{d}) + O(1)\,, \tag{3.15}$$

where

$$F^*(P,\, \vec{d}) = \frac{P^4}{d_1d_2d_3d_4} \sum_{\substack{0<|m_i|\leq H\\ 1,2,3,4}} c(m_i) \sum_{q\leq P} \frac{1}{q^4} \sum_{a\,(q)}{}^{*} e\Big(-\frac{aN}{q}\Big)\times$$
$$\times \sum_{|n_i-m_id_iq\eta|<M_i} G(q,\, ad_i^2,\, \vec{n}) \int_{\mathcal{N}(a,q)} J\Big(\beta P^2,\, \big(\vec{m}\eta - \frac{\vec{n}}{\vec{d}q}\big)P\Big) e(-\gamma)d\gamma.$$

Using Lemma 2.5 and working as in the proof of [14, Lemma 2] we find that

$$F^*(P,\, \vec{d}) = F^{'}(P,\, \vec{d}) + O\big(P^{3/2+\varepsilon}\big), \tag{3.16}$$

where

$$F^{'}(P,\, \vec{d}) = \frac{P^2}{d_1d_2d_3d_4} \sum_{\substack{0<|m_i|\leq H\\ i=1,2,3,4}} c(m_i) \sum_{q\leq P} \frac{1}{q^4} \sum_{\substack{|n_i-m_id_iq\eta|<M_i\\ (q,\,d_i)|n_i,\, i=1,\dots,4}} V_q(N,\, \vec{d}, 0,\, \vec{n})\times$$
$$\times \int_{|\gamma|\leq \frac{P}{2q}} J\Big(\gamma,\, \big(\vec{m}\eta - \frac{\vec{n}}{\vec{d}q}\big)P\Big) e(-\gamma)d\gamma,$$

and $V_q(N,\, \vec{d}, 0,\, \vec{n})$ is defined by (2.9). We represent the sum $F^{'}(P,\, \vec{d})$ as

$$F^{'}(P,\, \vec{d}) = F_1 + F_2\,, \tag{3.17}$$

where F_1 is the contribution of these addends with $q \leq Q$ and F_2 for addends with $Q < q \leq P$. Here Q is parameter, which we choose later. Using Lemma 2.3 (2), Lemma 2.6 and (3.1), we get

$$F_2 \ll \frac{P^2\delta^4}{d_1d_2d_3d_4} \sum_{\substack{0<|m_i|\leq H\\ i=1,2,3,4}} \sum_{Q<q\leq P} \frac{q^{5/2}\tau(q)(q,\, N)^{1/2}(q,\, d_1)...(q,\, d_4)}{q^4}\times$$
$$\times \sum_{\substack{|n_i-m_id_iq\eta|<M_i\\ (q,\,d_i)|n_i,\, i=1,\dots,4}} 1. \tag{3.18}$$

It is clear that the sum over $\vec{n}$ in the expression above is

$$\ll \prod_{1\leq i\leq 4} \sum_{\frac{-M_i+m_id_iq\eta}{(q,\,d_i)}<t_i<\frac{M_i+m_id_iq\eta}{(q,\,d_i)}} 1 \ll \frac{M_1M_2M_3M_4}{(q,d_1)(q,d_2)(q,d_3)(q,d_4)}$$
$$\ll \frac{P^{\varepsilon}d_1d_2d_3d_4}{(q,d_1)(q,d_2)(q,d_3)(q,d_4)}\,,$$

which, together with (3.18) and (3.2), gives

$$F_2 \ll P^{2+\varepsilon} \sum_{Q<q\le P} \frac{\tau(q)(q,\, N)^{1/2}}{q^{3/2}}.$$

Now we apply Cauchy's inequality to get

$$\begin{aligned} F_2 &\ll P^{2+\varepsilon} \Bigg(\sum_{Q<q\le P} \frac{\tau^2(q)}{q} \Bigg)^{\frac{1}{2}} \Bigg(\sum_{Q<q\le P} \frac{(q,N)}{q^2} \Bigg)^{\frac{1}{2}} \\ &\ll P^{2+\varepsilon} \Bigg(\sum_{\substack{t\mid N\\ t\le P}} t \sum_{\frac{Q}{t}<q_1\le\frac{P}{t}} \frac{1}{t^2 q_1^2} \Bigg)^{\frac{1}{2}} \ll \frac{P^{2+\varepsilon}}{Q^{1/2}}. \end{aligned} \tag{3.19}$$

To evaluate F_1 we firstly apply Lemma 2.4 to get

$$\int_{|\gamma|\le \frac{P}{2q}} \left| J\left(\gamma, (m\vec{\eta} - \frac{\vec{n}}{\vec{d}q})P\right)\right| d\gamma \ll \left(\left|(m\vec{\eta} - \frac{\vec{n}}{\vec{d}q})P\right|\right)^{-1+\varepsilon}.$$

Then using Lemma 2.6 and (3.2) we obtain

$$\begin{aligned} F_1 \ll \frac{P^2}{d_1d_2d_3d_4} &\sum_{q\le Q} \frac{q^{5/2}\tau(q)(q,\, N)^{1/2}(q,\, d_1)...(q,\, d_4)}{q^4}\times \\ &\times \sum_{\substack{|n_i-m_id_iq\eta|<M_i\\ (q,\, d_i)|n_i,\, i=1,\dots,4}} \frac{1}{\left|(\vec{m}\eta - \frac{\vec{n}}{\vec{d}q})P\right|}. \end{aligned} \tag{3.20}$$

It is clear that if $n_i = (q,\, d_i)t_i$, $d_i = (q,\, d_i)d_i'$ and

$$\left|(m_i\eta - \frac{n_i}{d_iq})P\right| = \frac{P(q,\, d_i)}{qd_i}|t_i - m_id_i'\eta q|\,,$$

then the sum over $(\vec{m}\eta - \frac{\vec{n}}{\vec{d}q})P$ in the expression above is

$$\ll \frac{q}{P} \sum_{|t_i-m_id_i'q\eta|<\frac{M_i}{(q,\, d_i)}} \frac{1}{\max\limits_{1\le i\le 4}(q,\, d_i)|t_i - m_id_i'\eta q|/d_i}. \tag{3.21}$$

Let t_1^o is such that

$$|t_1^o - m_1d_1'\eta q| = ||-m_1d_1'\eta q|| = ||m_1d_1'\eta q||\,.$$

As η is quadratic irrational number, then $||m_1d_1'\eta q|| \neq 0$ and for $t_1 \neq t_1^o$ we have $|t_1 - m_1d_1'\eta q| \ge 1/2$. Hence

$$\max_{1\le i\le 4} \frac{(q,\, d_i)|t_i - m_id_i'\eta q|}{d_1} \gg \frac{(q,\, d_1)}{d_1}\,,$$

which, together with (3.21), gives

$$\begin{aligned}
&\frac{q}{P}\sum_{|t_i-m_id_i'q\eta|<\frac{M_i}{(q,\,d_i)}}\frac{1}{\max\limits_{1\le i\le 4}(q,\,d_i)|t_i-m_id_i'\eta q|/d_i}\\
&\quad\ll\frac{q}{P}\biggl(\frac{d_1M_1M_2M_3M_4}{(q,\,d_1)^2(q,\,d_2)(q,\,d_3)(q,\,d_4)}+\frac{d_1M_2M_3M_4}{(q,\,d_1)(q,\,d_2)(q,\,d_3)(q,\,d_4)||m_1d_1'\eta q||}\biggr)\\
&\quad\ll\frac{qP^{\varepsilon-1}Dd_1d_2d_3d_4}{(q,\,d_1)^2(q,\,d_2)(q,\,d_3)(q,\,d_4)}+\frac{qP^{\varepsilon-1}d_1d_2d_3d_4}{(q,\,d_1)(q,\,d_2)(q,\,d_3)(q,\,d_4)||m_1d_1'\eta q||}. \qquad (3.22)
\end{aligned}$$

As η is quadratic irrationality, it has periodic continued fraction and if $\frac{a_n}{b_n}$, $n\in\mathbb{N}$ is the n-th convergent, then $b_n\le c^n$ for some constant $c>0$. Using that $||m_1d_1'q||\le\frac{HDQ}{(d_1,\,q)}$ and Liouville's inequality for quadratic numbers (see Lemma 2.7), we can find convergent $\frac{a}{b}$ to η with denominator such that

$$\frac{3HDQ}{(d_1,\,q)}<b\ll_c\frac{HDQ}{(d_1,\,q)}\,. \qquad (3.23)$$

Since $(a,\,b)=1$ we have that $m_1d_1'q\frac{a}{b}\notin\mathbb{Z}$. As $\left|\eta-\frac{a}{b}\right|<\frac{1}{b^2}$ and (3.23) we get

$$\begin{aligned}
||m_1d_1'q\eta||&\ge\left|\left|m_1d_1'q\frac{a}{b}\right|\right|-\left|\left|m_1d_1'q\Bigl(\eta-\frac{a}{b}\Bigr)\right|\right|\ge\left|\left|m_1d_1'q\frac{a}{b}\right|\right|-\frac{|m_1|d_1'q}{b^2}\\
&>\frac{1}{b}-\frac{|m_1|d_1'q(d_1,\,q)}{3bHDQ}\ge\frac{1}{b}-\frac{|m_1|d_1q}{3bHDQ}\\
&>\frac{1}{b}-\frac{|m_1|}{3bH}\ge\frac{1}{b}-\frac{1}{3b}=\frac{2}{3b}\\
&\gg\frac{(d_1,\,q)}{HDQ}\,.
\end{aligned}$$

From (3.21) and (3.22) it follows that

$$\sum_{\substack{|n_i-m_id_iq\eta|<M_i\\(q,\,d_i)|n_i,\,i=1,\dots,4}}\frac{1}{\bigl|(\vec{m}\eta-\frac{\vec{n}}{\vec{d}q})P\bigr|}\ll\frac{qP^{\varepsilon-1}d_1d_2d_3d_4HDQ}{(q,\,d_1)^2(q,\,d_2)(q,\,d_3)(q,\,d_4)}\,.$$

Then for F_1 (see (3.20)) we obtain

$$F_1\ll\frac{P^{1+\varepsilon}DQ}{\delta}\sum_{q\le Q}\frac{\tau(q)(q,\,N)^{1/2}}{q^{1/2}}. \qquad (3.24)$$

Applying Cauchy's inequality we get

$$
\begin{aligned}
F_1 &\ll \frac{P^{1+\varepsilon}DQ}{\delta}\left(\sum_{q\le Q}\tau^2(q)\right)^{\frac12}\left(\sum_{q\le Q}\frac{(q,N)}{q}\right)^{\frac12}\\
&\ll \frac{P^{1+\varepsilon}DQ}{\delta}\cdot Q^{1/2}(\log Q)^{3/2}\left(\sum_{\substack{t|N\\ t\le Q}}\sum_{q_1\le\frac{Q}{t}}\frac{1}{q_1}\right)^{\frac12}\\
&\ll \frac{P^{1+\varepsilon}DQ^{3/2}}{\delta}.
\end{aligned}
\tag{3.25}
$$

We choose $Q=\delta^{1/2}P^{1/2}D^{-1/2}$. Then

$$F_1,\, F_2 \ll P^{7/4+\varepsilon}\delta^{-1/4}D^{1/4}\,.$$

From (3.14), (3.15), (3.16), (3.17) it follows that

$$\Gamma_1^* \ll D^{17/4}P^{7/4+\varepsilon}\delta^{-1/4}\,.$$

The estimate of Γ_5^* goes along the same lines.

3.4. END OF THE PROOF OF THEOREM 1.1

From (3.10) and (3.11) we get

$$\Gamma \gg \frac{\delta N}{(\log N)^4} + D^{17/4}P^{7/4+\varepsilon}\delta^{-1/4}\,.$$

Then for a fixed small $\varepsilon>0$, $\lambda<\frac{1-8\varepsilon}{10}$, $D<N^{\frac{1-10\lambda-8\varepsilon}{34}}$ and $z=D^{1/3,13}$ we get $\Gamma\gg\frac{\delta N}{(\log N)^4}$. So the equation (1.1) have solutions in almost-prime numbers $x_1,\dots,x_4\in\mathcal{P}_k$, $k=\left[\frac{53,21}{1-10\lambda-8\varepsilon}\right]$ such that $\{\eta x_i\}<N^{-\lambda}$, $i=1,2,3,4$.

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